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Noncommutative tori and their crossed products by finite groups

(비가환 원환체와 유한군에 의한 교차곱)

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Noncommutative tori and their crossed products by finite groups

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by

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Abstract

Noncommutative tori and their crossed products by finite groups

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We study canonical group actions on higher dimensional noncommutative tori which generalize the $\mathrm{SL}_2(\mathbb{Z})$ actions on 2-dimensional noncommutative tori, namely, rotation algebras and show that the crossed products of higher dimensional simple tori by canonical actions of finite groups are not necessarily AF algebras but AT algebras. This result on non-AF crossed products is contrasted with the recently known fact that the crossed products of irrational rotation algebras by finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are always AF aglebras.

Key words: Noncommutative torus, Group action, C^*-crossed product, AF algebra
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Contents

Chapter 1

Introduction

1.1 $\mathrm{SL}_2(\mathbb{Z})$ actions on rotation algebras

The rotation algebra $\mathcal{A}_\theta$, $\theta \in \mathbb{R}$, is the universal C^*-algebras generated by two unitaries u_1, u_2 subject to the commuting relation $u_2u_1 = \exp(2\pi i\theta)u_1u_2$. If θ is an integer, $\mathcal{A}_\theta$ is isomorphic to the commutative C^*-algebra $C(\mathbb{T}^2)$ of all continuous functions on 2-dimensional torus $\mathbb{T}^2$. So the rotation algebras are also called the 2-dimensional noncommutative tori. It is well-known that a rotation algebra $\mathcal{A}_\theta$ is simple if and only if θ is an irrational number. We call this simple $\mathcal{A}_\theta$ an irrational rotation algebra.

2-dimensional noncommutative tori can be generalized to higher dimensional noncommutative tori in such a way that one allows more unitary generators which commute up to scalars. More precisely, for a given $d\times d$ ($d \geq 2$) real skew symmetric matrix $\Theta = (\theta_{jk})$, the d-dimensional noncommutative torus $\mathcal{A}_\Theta$ is defined to be the universal C^*-algebra generated by d unitaries $u_1, \ldots, u_d$ satisfying the relations

$$u_ju_k = \exp(2\pi i\theta_{kj})u_ku_j$$

for $k, j = 1, \ldots, d$. Apparently, every rotation algebra $\mathcal{A}_\theta$ can be considered as a 2-dimensional noncommutative torus $\mathcal{A}_\Theta$ associated with $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$.

The rotation algebra $\mathcal{A}_\theta$ is known to be isomorphic to the crossed product $C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$, where α is the natural action of $\mathbb{Z}$ on $C(\mathbb{T})$ induced by the homeomorphism $z \mapsto \exp(2\pi i\theta)z : \mathbb{T} \to \mathbb{T}$ rotating $\mathbb{T}$ by the angle $2\pi\theta$. This explains why we call $\mathcal{A}_\theta$ a rotation algebra. In 1980, Pimsner and Voiculescu proved the short exact sequence of K-groups, called PV-sequence, in order to compute K-groups for the crossed products by $\mathbb{Z}$ actions and using the PV-sequence they showed that for an irrational rotation algebra $\mathcal{A}_\theta$,

$$K_1(\mathcal{A}_\theta) \cong \mathbb{Z} \oplus \mathbb{Z}$$

as abelian groups with generators $[u_1]$, $[u_2]$ and

$$(K_0(\mathcal{A}_\theta), K_0(\mathcal{A}_\theta)^+, [1]_0) \cong (\mathbb{Z} + \theta\mathbb{Z}, (\mathbb{Z} + \theta\mathbb{Z}) \cap \mathbb{R}^+, 1)$$

as ordered abelian groups with distinguished order units [20]. From this it follows that

$$\mathrm{Aut}(K_0(\mathcal{A}_\theta), K_0(\mathcal{A}_\theta)^+, [1]_0) = \{\mathrm{id}\}, \quad \mathrm{Aut}(K_1(\mathcal{A}_\theta)) = \mathrm{GL}_2(\mathbb{Z}).$$

Thus one might expect that for each matrix $A \in \mathrm{GL}_2(\mathbb{Z}) = \mathrm{Aut}(K_1(\mathcal{A}_\theta))$ there exists an automorphism ψ on $\mathcal{A}_\theta$ such that $K_1(\psi) = A$, and this actually turned out to be true due to the work by G. A. Elliott and D. E. Evans. In [6], it was shown that every irrational rotation algebra is an AT algebra of real rank zero hence it is classifiable by its K-theory which implies that any automorphism A at the level of K-groups can be lifted to an automorphism ψ at the level of algebras so that $K_*(\psi) = A$.

In 1981, Y. Watatani considered an action $\alpha : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{A}_\theta)$ of the group $\mathrm{SL}_2(\mathbb{Z})$ on rotation algebras $\mathcal{A}_\theta$ which is given by

$$\begin{aligned}\alpha_A(u_1) &= \exp(\pi i\theta a_{11}a_{21})u_1^{a_{11}}u_2^{a_{21}},\\ \alpha_A(u_2) &= \exp(\pi i\theta a_{12}a_{22})u_1^{a_{12}}u_2^{a_{22}}\end{aligned} \tag{1.1}$$

for $A = (a_{jk}) \in \mathrm{SL}_2(\mathbb{Z})$ [26], and classified these automorphisms α_A, ($A \in \mathrm{SL}_2(\mathbb{Z})$) using the notion of K_1-entropy. B. A. Brenken [2] used the same automorphism to study representations of rotation algebras. In fact, the automorphism α_A satisfies $K_1(\alpha_A) = A$ for every $A \in \mathrm{SL}_2(\mathbb{Z})$. In this thesis, actions of $\mathrm{SL}_2(\mathbb{Z})$ on rotation algebras will refer to these actions $\alpha : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{A}_\theta)$.

1.2 Structure of the crossed products of irrational rotation algebras by finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$

The group $\mathrm{SL}_2(\mathbb{Z})$ is known to have only four nontrivial finite subgroups up to conjugacy which are the cyclic groups $\mathbb{Z}_n$ of orders $n = 2, 3, 4$ and 6. Each $\mathbb{Z}_n (\subset \mathrm{SL}_2(\mathbb{Z}))$ is generated by the following matrix $A_n \in \mathrm{SL}_2(\mathbb{Z})$ of order n:

$$A_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

By restricting the action of $SL_2(\mathbb{Z})$ given in (1.1) to these finite subgroups $\mathbb{Z}_n$, we obtain finite group actions

$$\alpha : \mathbb{Z}_n \to \mathrm{Aut}(\mathcal{A}_\theta) \tag{1.2}$$

on $\mathcal{A}_\theta$ for $n = 2, 3, 4, 6$.

There have been a great deal of study on the crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_n$ and the fixed point subalgebras $\mathcal{A}_\theta^{\mathbb{Z}_n}$ of irrational rotation algebras $\mathcal{A}_\theta$ by finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$. In particular, the crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_2$ are very well understood: they are known to be AF algebras in [1] and their K-groups are computed in [12]. The crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_4$ by $\mathbb{Z}_4$ has also been proved to be AF algebras for most irrational numbers θ by S. Walters in [25]. The fixed point subalgebra $\mathcal{A}_\theta^{\mathbb{Z}_n}$ has been intensively studied by C. Farsi and N. Watling in [7] where they also classified the fixed point algebras by an automorphism α_A for $A \in \mathrm{SL}_2(\mathbb{Z})$ which may not be of finite order. In a subsequence paper [8], they characterized the fixed point subalgebras $\mathcal{A}_\theta^{\mathbb{Z}_4}$ and computed the K-groups when θ is rational, and then continued their work for $n = 3$ in [9] and for $n = 6$ in [10].

In 2010, the most culminating work along the same line was done by S. Echterhoff, W. Lück, N. C. Phillips and S. Walters. In [4] they showed that the crossed product of irrational rotation algebras $\mathcal{A}_\theta$ by the action of $\mathbb{Z}_n$ given in (1.2) are all AF algebras and the same is true for the fixed point subalgebras $\mathcal{A}_\theta^{\mathbb{Z}_n}$. They also considered the *flip action* α by $\mathbb{Z}_2$ on higher dimensional noncommutative simple tori. It is the action generated by the *flip automorphism* $\sigma \in \mathrm{Aut}(\mathcal{A}_\Theta)$ of order 2 which is defined by

$$\sigma(u_j) = u_j^*, \tag{1.3}$$

where u_j's are the unitary generators of $\mathcal{A}_\Theta$. For 2-dimensional tori, the flip action α is the action by $\mathbb{Z}_2 (\subset \mathrm{SL}_2(\mathbb{Z}))$ generated by A_2. As in the case of 2-dimensional simple tori, the crossed products $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_2$ and the fixed point algebras $\mathcal{A}_\Theta^{\mathbb{Z}_2}$ are known to be AF algebras [4]. What they have done in [4] to prove the crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_n$ and the fixed point subalgebras $\mathcal{A}_\theta^{\mathbb{Z}_n}$ are AF, roughly consists of two steps: they first showed that the crossed products belong to the class of unital simple nuclear separable C^*-algebras of tracial rank zero which satisfy the Universal Coefficient Theorem. Secondly, they computed the K-theory of the crossed products which turned out to be same as the K-theory of suitable AF-algebras in the same class. Since the class mentioned above obeys a classification theorem with K-theory as a complete invariance due to H. Lin [16], they could show that the crossed products are AF. The fixed point subalgebras are automatically AF since they are isomorphic to hereditary subalgebras of crossed products [22] which are simple C^*-algebras; thus $\mathcal{A}_\theta^{\mathbb{Z}_n}$ and $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_n$ are stably isomorphic.

It would then be a very natural question to ask whether the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ of a simple higher dimensional noncommutative torus $\mathcal{A}_\Theta$ is still AF when α is an action of a finite subgroup G of $\mathrm{SL}_d(\mathbb{Z})$ (or $\mathrm{GL}_d(\mathbb{Z})$). But it was unclear even whether there are any known finite groups acting on some higher dimensional noncommutative simple tori in a canonical way, except the flip action by $\mathbb{Z}_2$. Thus the purpose of this thesis is two-fold. One is to find finite subgroups G of $\mathrm{GL}_d(\mathbb{Z})$ which act canonically on higher dimensional noncommutative simple d-tori $\mathcal{A}_\Theta$ and the other is to figure out if the crossed products $\mathcal{A}_\Theta \rtimes_\alpha G$ are AF when they are simple. In this thesis we answer this question by providing finite group actions on higher dimensional simple tori which give rise to non-AF crossed products.

This thesis is based on [11].

1.3 Brief overview of the thesis

In this section we briefly overview what will follow in this thesis.

There is no doubt that a desirable candidate for actions on d-dimensional noncommutative torus $\mathcal{A}_\Theta$ must extend the $\mathrm{SL}_2(\mathbb{Z})$ action on rotational algebra and contain considerably many interesting examples of finite group actions such as the flip action. In Chapter 2, we obtain such an action which we will call a *canonical action*. For this we need to identify $\mathcal{A}_\Theta$ with the twisted group algebra $C^*(\mathbb{Z}^d, \omega_\Theta)$ twisted by a suitable 2-cocycle ω_Θ on $\mathbb{Z}^d$ via the isomorphism $u_j \mapsto l_\Theta(e_j)$, where $l_\Theta(e_j)$ is the image of jth standard basis element e_j of $\mathbb{Z}^d$ under the regular ω_Θ-representation l_Θ of $\mathbb{Z}^d$.

Theorem 1.3.1. (Theorem 2.3.5) *Let $\mathcal{A}_\Theta$ be the d-dimensional noncommutative torus associated with Θ and let*

$$G_\Theta = \{A \in \mathrm{GL}_d(\mathbb{Z}) : \Theta = A^t \Theta A\}$$

be the subgroup of $\mathrm{GL}_d(\mathbb{Z})$, where A^t denotes the transpose of A. Then the map $\alpha : G_\Theta \to \mathrm{Aut}(\mathcal{A}_\Theta)$ given by $\alpha_A(l_\Theta(y)) = l_\Theta(Ay)$ for $A \in G_\Theta$ and $y \in \mathbb{Z}^d$ is a group action which reduces to the $\mathrm{SL}_2(\mathbb{Z})$ action if $\mathcal{A}_\Theta$ is a 2-dimensional rotation algebra.

In contrast with the 2-dimensional case, the group G_Θ is a subgroup of $\mathrm{GL}_d(\mathbb{Z})$ instead of $\mathrm{SL}_d(\mathbb{Z})$ so that the flip, generated by the matrix $-I_d$, can also be covered as an action considered in the above theorem. Note here that the group G_Θ depends on Θ, hence on the torus $\mathcal{A}_\Theta$.

Chapter 3 starts with finding finite cyclic groups generated by matrices A of finite order. Then we will try to seek all the Θ's such that the group G_Θ contains the matrix A. Actually, this way of examining $\mathcal{A}_\Theta$ with finite groups already given or discovered turns to be more practical than starting with $\mathcal{A}_\Theta$ to find finite group actions on $\mathcal{A}_\Theta$. For a given dimension $d \geq 2$ if one knows all the matrices of finite order up to conjugacy in $\mathrm{GL}_d(\mathbb{Z})$, then our strategy guarantees that one can also know all the possible Θ's which admit canonical actions by finite cyclic groups on $\mathcal{A}_\Theta$. In Chapter 3, we will actually do this in the case of $d = 3$ due to the list of all the elements of finite order up to conjugacy established in [24] and obtain the following.

Theorem 1.3.2. (Theorem 3.3.1) *The only canonical action by a nontrivial* *finite cyclic group on a simple* 3*-dimensional torus is the flip action.*

In Chapter 4, we devote to study the structure of the crossed products of simple tori by canonical actions of finite groups. Suppose $\alpha : G \to \mathrm{Aut}(\mathcal{A}_\Theta)$ is a canonical action of a finite subgroup G of G_Θ on a d-dimensional noncommutative simple torus $\mathcal{A}_\Theta$. Following the steps in [4], we show that the resulting crossed products also belong to the class of all unital simple nuclear separable C^*-algebras of tracial rank zero which satisfy the Universal Coefficient Theorem. For K-theory we use the method developed in [4] to show that the K-theory of the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G (= C^*(\mathbb{Z}^d \rtimes G, \widetilde{\omega}_\Theta))$ is isomorphic to the K-theory of the (untwisted) group C^*-algebra $C^*(\mathbb{Z}^d \rtimes G)$ of the semidirect product group $\mathbb{Z}^d \rtimes G$, where G acts on $\mathbb{Z}^d$ as a conjugation action. If the conjugation action is *free outside the origin*, that is, the action of G on $\mathbb{Z}^d \setminus \{0\}$ is free, then we can use the formulas for topological K-theory of group C^*-algebras $C^*(\mathbb{Z}^d \rtimes \mathbb{Z}_n)$ obtained recently in [14, 3].

In order to find elements in $\mathrm{GL}_d(\mathbb{Z})$ of finite order in this thesis, we focus on the companion matrices of cyclotomic polynomials. The companion matrix C_n of the nth cyclotomic polynomial with $n \geq 3$ is an integral matrix in $\mathrm{SL}_d(\mathbb{Z})$ of order n, where d is the number of all the distinct primitive nth roots of unity, so $d = \phi(n)$ for ϕ the Euler's totient function. Then we obtain a canonical action of $\mathbb{Z}_n = \langle C_n \rangle$ on a d-dimensional noncommutative torus $\mathcal{A}_\Theta$ with $C_n \in G_\Theta$. We also show that the corresponding conjugation action of $\mathbb{Z}_n = \langle C_n \rangle$ on $\mathbb{Z}^d$ is free outside the origin and then apply the formulas of [14, 3] to the semidirect product $\mathbb{Z}^d \rtimes \mathbb{Z}_n$ to obtain the following result which plays a key role in this thesis.

Theorem 1.3.3. (Theorem 4.2.1, Theorem 4.2.11) *Let $n \geq 3$ be an integer with $d = \phi(n)$. Let $\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n$ be the semidirect product group of $\mathbb{Z}^d$ by the conjugation action α of $\mathbb{Z}_n = \langle C_n \rangle$ on $\mathbb{Z}^d$. If $2d \geq n+5$, then $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$ is nontrivial. If n is even, then $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$ is trivial. In the case* *of prime n, $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$ is trivial if and only if $n = 3, 5$.*

For the last step towards non-AF crossed products results, in Chapter 5, we show that there really do exist $\phi(n)$-dimensional noncommutative simple tori $\mathcal{A}_\Theta$ which admit the canonical actions of the cyclic group $\mathbb{Z}_n = \langle C_n \rangle$. Therefore we obtain the main result of this thesis which can be stated as follows:

Theorem 1.3.4. (Theorem 5.1.3) *Let $n \geq 3$ be an integer with $d = \phi(n)$. Then there exists a canonical action $\alpha : \mathbb{Z}_n \to \mathrm{Aut}(\mathcal{A}_\Theta)$ of $\mathbb{Z}_n = \langle C_n \rangle$ on a simple d-dimensional noncommutative torus $\mathcal{A}_\Theta$ and the crossed products $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is an AT algebra. Furthermore,*

(i) *If $2d \geq n + 5$ holds, then the crossed product is not an AF algebra.*

(ii) *If n is even, then the crossed product is an AF algebra.*

(iii) *If n is odd prime, the crossed product is an AF algebra if and only if $n = 3, 5$.*

We also describe a general form of Θ to which a simple $\phi(n)$-dimensional noncommutative torus $\mathcal{A}_\Theta$ is associated so that $\mathcal{A}_\Theta$ admits a canonical action of the finite group $\mathbb{Z}_n$ generated by C_n. Finally we give an interesting example of canonical actions on 4-dimensional simple torus $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ which is the 2-fold tensor product of an irrational rotation algebra $\mathcal{A}_\theta$ with itself.

Chapter 2

Noncommutative tori and Canonical actions

In this chapter, we define a canonical action on noncommutative tori using the fact that noncommutative tori can be viewed as twisted group C^*-algebras.

2.1 Twisted group algebras of discrete groups

Let G be a second countable discrete group. A *2-cocycle* on G is a function $\omega : G \times G \to \mathbb{T}$ such that $\omega(x,y)\omega(xy,z) = \omega(y,z)\omega(x,yz)$ and $\omega(x,1) = \omega(1,x) = 1$ for $x,y,z \in G$. Then the *twisted convolution algebra* $\ell^1(G,\omega)$ is defined to be the involutive algebra of all summable functions on G with the involution and the convolution given by

$$(f *_\omega g)(x) = \sum_{y\in G} f(y)g(y^{-1}x)\omega(y,y^{-1}x),$$
$$f^*(x) = \overline{\omega(x,x^{-1})f(x^{-1})}$$

for $x \in G$. We call a map $v : G \to \mathcal{U}(\mathcal{H})$ of G into the unitary group of a Hilbert space $\mathcal{H}$ an *ω-representation* of G if

$$v(x)v(y) = \omega(x,y)v(xy) \tag{2.1}$$

for $x, y \in G$. The *regular ω-representation* of G is the ω-representation $l_\omega : G \to \mathcal{U}(\ell^2(G))$ given by

$$(l_\omega(x)\xi)(y) = \omega(x, x^{-1}y)\xi(x^{-1}y)$$

for $\xi \in \ell^2(G)$ and $x, y \in G$. Every ω-representation $v : G \to \mathcal{U}(\mathcal{H})$ induces a contractive $*$-homomorphism $v : \ell^1(G,\omega) \to B(\mathcal{H})$ (also denoted v) given by $v(f) = \sum_x f(x)v(x)$ for $f \in \ell^1(G,\omega)$, and every nondegenerate representation of $\ell^1(G,\omega)$ arises in this way. The *full twisted group algebra* $C^*(G,\omega)$ is then defined to be the enveloping C^*-algebra of $\ell^1(G,\omega)$ which can be characterized by the unique C^*-algebra having the universal property in the sense of that any $*$-homomorphism $\pi : \ell^1(G,\omega) \to B(\mathcal{H})$ can be extended uniquely to a $*$-homomorphism $\overline{\pi} : C^*(G,\omega) \to B(\mathcal{H})$ satisfying $\overline{\pi} \circ \iota = \pi$ where $\iota : \ell^1(G,\omega) \to C^*(G,\omega)$ is the natural embedding. Then the *reduced twisted group algebra* $C^*_r(G,\omega)$ is defined to be the image of $C^*(G,\omega)$ under $\overline{l_\omega}$. So every reduced twisted group algebra $C^*_r(G,\omega)$ is contained as a C^*-subalgebra in $B(\ell^2(G))$. It is known that $C^*(G,\omega)$ is equal to $C^*_r(G,\omega)$ if G is amenable [27, Section 5]. In this case, from (2.1) and the previous explanation, we see that

$$C^*(G,\omega) = \overline{\mathrm{span}}\{l_\omega(x) : x \in G\}. \tag{2.2}$$

2.2 Noncommutative tori

Recall that a d-dimensional *noncommutative torus* $\mathcal{A}_\Theta$ associated with a $d\times d$ skew symmetric matrix $\Theta = (\theta_{kj})$ is defined to be the universal C^*-algebra generated by d unitaries $u_1, \ldots, u_d$ subject to the relations

$$u_j u_k = \exp(2\pi i \theta_{jk}) u_k u_j \tag{2.3}$$

for $j, k = 1, \ldots, d$.

But for the purpose of obtaining group actions, it is more convenient to use an alternative definition of noncommutative tori [21] given as twisted group algebras. It is known that two definition coincide even though one can hardly find a proof in the literature. In this section, we will show that the d-dimensional noncommutative torus $\mathcal{A}_\Theta$ associated with a $d \times d$ skew symmetric matrix Θ in the above sense coincides with the twisted group algebra $C^*(\mathbb{Z}^d, \omega_\Theta)$ twisted by a 2-cocycle $\omega_\Theta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$. Note that here we do not distinguish $C^*(\mathbb{Z}^d, \omega_\Theta)$ from $C^*_r(\mathbb{Z}^d, \omega_\Theta)$ since the group $\mathbb{Z}^d$ is amenable. Also we introduce a known criterion on simplicity of a noncommutative torus $\mathcal{A}_\Theta$ or $C^*(\mathbb{Z}^d, \omega_\Theta)$ in terms of Θ. We start from setting up notations which will be frequently used in this thesis.

Notations 2.2.1. Let $d \geq 2$ be a positive integer.

(i) We set

$$\mathcal{T}_d(\mathbb{R}) := \{\Theta \in \mathbb{M}_d(\mathbb{R}) : \Theta^t = -\Theta\},$$

where Θ^t denotes the transpose of Θ. Similarly, $\mathcal{T}_d(\mathbb{Z})$ denotes the set of all $d \times d$ skew symmetric matrices with entries from $\mathbb{Z}$.

(ii) For $\Theta \in \mathcal{T}_d(\mathbb{R})$, let $\omega_\Theta : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$ be the 2-cocycle defined by

$$\omega_\Theta(x, y) = \exp(\pi i \langle \Theta x, y \rangle) \tag{2.4}$$

for $x, y \in \mathbb{Z}^d$, where $\langle \cdot, \cdot \rangle$ denotes the standard pairing of $\mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}$.

(iii) For $\Theta \in \mathcal{T}_d(\mathbb{R})$, let l_Θ denote the regular ω_Θ-representation l_{ω_Θ}.

Lemma 2.2.2. *Let $(e_j)_{j=1}^d$ be the standard basis of $\mathbb{Z}^d$ and $\Theta = (\theta_{jk}) \in \mathcal{T}_d(\mathbb{R})$ be given. Then we have the following.*

(i) $l_\Theta(e_j)l_\Theta(e_k) = \exp(2\pi i\theta_{kj})l_\Theta(e_k)l_\Theta(e_j)$ *holds for each* $k, j = 1, \ldots, d$.

(ii) *For* $y = \sum_{j=1}^d y_j e_j \in \mathbb{Z}^d$, *let* $c_\Theta(y) \in \mathbb{T}$ *be the scalar given by*

$$c_\Theta(y) = \exp(\pi i \sum_{k=2}^{d} \sum_{j=1}^{k-1} y_k y_j \theta_{jk}). \tag{2.5}$$

Then

$$l_\Theta(y) = c_\Theta(y) l_\Theta(e_1)^{y_1} \cdots l_\Theta(e_d)^{y_d}.$$

(iii) *The map* $\psi : \mathbb{Z}^d \to \mathcal{A}_\Theta$ *defined by*

$$\psi(y) = c_\Theta(y) u_1^{y_1} \ldots u_d^{y_d}$$

is a ω_Θ*-representation of* $\mathbb{Z}^d$.

Proof. (i) Since l_Θ is an ω_Θ-representation, by (2.1) with the abelian group operation of $\mathbb{Z}^d$ it is easy to see that for $x, y \in \mathbb{Z}^d$,

$$\begin{aligned} l_\Theta(x)l_\Theta(y) &= \omega_\Theta(x, y) l_\Theta(x + y) \\ &= \omega_\Theta(x, y) l_\Theta(y + x) \\ &= \omega_\Theta(x, y)\overline{\omega_\Theta(y, x)} l_\Theta(y) l_\Theta(x) \\ &= \omega_\Theta(x, y)^2 l_\Theta(y) l_\Theta(x). \end{aligned}$$

In particular, we have for $j, k = 1, \ldots, d$,

$$l_\Theta(e_j)l_\Theta(e_k) = \omega_\Theta(e_j, e_k)^2 l_\Theta(e_k)l_\Theta(e_j) = \exp(2\pi i\theta_{kj})l_\Theta(e_k)l_\Theta(e_j).$$

(ii) Note that

$$l_\Theta(x+y) = \omega_\Theta(x,y)^{-1}l_\Theta(x)l_\Theta(y)$$

for $x, y \in \mathbb{Z}^d$ by (2.1). Also note that $\omega_\Theta(x, \lambda x) = \exp(\lambda\pi i\langle\Theta x, x\rangle) = 1$ for all $x \in \mathbb{Z}^d$ and $\lambda \in \mathbb{R}$. Therefore recursively we have

$$\begin{aligned}
l_\Theta(y) &= l_\Theta(\sum_{j=1}^d y_je_j) \\
&= l_\Theta(y_1e_1 + \sum_{j=2}^d y_je_j) \\
&= \omega_\Theta(y_1e_1, \sum_{j=2}^d y_je_j)^{-1} l_\Theta(y_1e_1)l_\Theta(\sum_{j=2}^d y_je_j) \\
&= \omega_\Theta(y_1e_1, \sum_{j=2}^d y_je_j)^{-1}\omega_\Theta(y_2e_2, \sum_{j=3}^d y_je_j)^{-1} l_\Theta(y_1e_1)l_\Theta(y_2e_2)l_\Theta(\sum_{j=3}^d y_je_j) \\
&= \dots \\
&= \Big[\prod_{k=1}^{d-1}\omega_\Theta(y_ke_k, \sum_{j=k+1}^d y_je_j)^{-1}\Big] l_\Theta(y_1e_1)\cdots l_\Theta(y_de_d).
\end{aligned}$$

Then by the definition of ω_Θ given in (2.4), we have

$$\begin{aligned}
l_\Theta(y) &= \exp(-\pi i \sum_{k=1}^{d-1} \langle \Theta y_k e_k, \sum_{j=k+1}^{d} y_j e_j \rangle) l_\Theta(y_1 e_1) \cdots l_\Theta(y_d e_d) \\
&= \exp(-\pi i \sum_{k=1}^{d-1} \sum_{j=k+1}^{d} y_k y_j \theta_{jk}) l_\Theta(e_1)^{y_1} \cdots l_\Theta(e_d)^{y_d} \\
&= \exp(\pi i \sum_{k=1}^{d-1} \sum_{j=k+1}^{d} y_k y_j \theta_{kj}) l_\Theta(e_1)^{y_1} \cdots l_\Theta(e_d)^{y_d} \\
&= \exp(\pi i \sum_{j=1}^{d-1} \sum_{k=j+1}^{d} y_k y_j \theta_{jk}) l_\Theta(e_1)^{y_1} \cdots l_\Theta(e_d)^{y_d} \\
&= \exp(\pi i \sum_{k=2}^{d} \sum_{j=1}^{k-1} y_k y_j \theta_{jk}) l_\Theta(e_1)^{y_1} \cdots l_\Theta(e_d)^{y_d} \\
&= c_\Theta(y) l_\Theta(e_1)^{y_1} \cdots l_\Theta(e_d)^{y_d}.
\end{aligned}$$

(iii) It is enough to show that

$$\begin{aligned}
&u_1^{x_1} \cdots u_d^{x_d} u_1^{y_1} \cdots u_d^{y_d} \\
= \ &\omega_\Theta(x,y) c_\Theta(x)^{-1} c_\Theta(y)^{-1} c_\Theta(x+y) u_1^{x_1+y_1} \cdots u_d^{x_d+y_d}
\end{aligned}$$

holds for $x = \sum x_j e_j$, $y = \sum y_j e_j \in \mathbb{Z}^d$. Using (2.3) we see that

$$\begin{aligned}
u_1^{x_1} \cdots u_d^{x_d} u_1^{y_1} \cdots u_d^{y_d} &= \prod_{k=1}^{d-1} \prod_{j=k+1}^{d} \exp(2\pi i x_j y_k \theta_{kj}) u_1^{x_1+y_1} \cdots u_d^{x_d+y_d} \\
&= \exp(2\pi i \sum_{k=1}^{d-1} \sum_{j=k+1}^{d} x_j y_k \theta_{kj}) u_1^{x_1+y_1} \cdots u_d^{x_d+y_d}.
\end{aligned}$$

Then the following computation proves (iii).

$$\begin{aligned}
&\omega_\Theta(x,y)c_\Theta(x)^{-1}c_\Theta(y)^{-1}c_\Theta(x+y)\\
&=\exp(\pi i\sum_{k,j=1}^{d} x_j y_k\theta_{kj})\exp(\pi i\sum_{k=2}^{d}\sum_{j=1}^{k-1}((x_k+y_k)(x_j+y_j)-x_kx_j-y_ky_j)\theta_{jk})\\
&=\exp(\pi i\Big[\sum_{k,j=1}^{d} x_j y_k\theta_{kj}+\sum_{k=2}^{d}\sum_{j=1}^{k-1}x_jy_k\theta_{jk}+\sum_{k=2}^{d}\sum_{j=1}^{k-1}x_ky_j\theta_{jk}\Big])\\
&=\exp(\pi i\Big[\sum_{k=1}^{d}\sum_{j=1}^{d} x_j y_k\theta_{kj}-\sum_{k=2}^{d}\sum_{j=1}^{k-1}x_jy_k\theta_{kj}+\sum_{j=2}^{d}\sum_{k=1}^{j-1}x_jy_k\theta_{kj}\Big])\\
&=\exp(2\pi i\sum_{k=1}^{d-1}\sum_{j=k+1}^{d} x_jy_k\theta_{kj}).
\end{aligned}$$

□

Now we are ready to prove the following which is well-known without proof in the literature.

Proposition 2.2.3. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$. Then the d-dimensional noncommutative torus $\mathcal{A}_\Theta$ associated with Θ is isomorphic to the twisted group algebra $C^*(\mathbb{Z}^d, \omega_\Theta)$ via an isomorphism Φ such that $\Phi(u_j) = l_\Theta(e_j)$.*

Proof. Lemma 2.2.2(i) ensures that by the universal property of $\mathcal{A}_\Theta$ there exists a $*$-homomorphism $\Phi : \mathcal{A}_\Theta \to C^*(\mathbb{Z}^d, \omega_\Theta)$ such that $\Phi(u_j) = l_\Theta(e_j)$. Since the map ψ in the Lemma 2.2.2(iii) is an ω_Θ representation of $\mathbb{Z}^d$ on $\mathcal{A}_\Theta$, it induces a contractive $*$-homomorphism $\ell^1(\mathbb{Z}^d, \omega_\Theta) \to \mathcal{A}_\Theta$ which in turn can be extended to a $*$-homomorphism $\Psi : C^*(\mathbb{Z}^d, \omega_\Theta) \to \mathcal{A}_\Theta$ such that $\Psi(l_\Theta(e_j)) = c_\Theta(e_j)u_j = u_j$. Since u_j's generate $\mathcal{A}_\Theta$ and $l_\Theta(e_j)$'s generate $C^*(\mathbb{Z}^d, \omega_\Theta)$, Φ and Ψ are mutually inverses. This proves the proposition.

□

Now we are in position to state an alternative definition of noncommutative tori.

Definition 2.2.4. Let $\Theta = (\theta_{jk}) \in \mathcal{T}_d(\mathbb{R})$ be a $d \times d$ skew symmetric matrix and ω_Θ be the 2-cocycle of $\mathbb{Z}^d$ given in (2.4). The *noncommutative torus* $\mathcal{A}_\Theta$ associated to Θ is defined to be the twisted group algebra $C^*(Z^d, \omega_\Theta)$ (or equivalently the universal C^*-algebra generated by d unitaries $u_1, \dots, u_d$ satisfying the relations $u_j u_k = \exp(2\pi i \theta_{kj}) u_k u_j$).

Remark 2.2.5. By the above definition, we regard $\mathcal{A}_\Theta = C^*(\mathbb{Z}^d, \omega_\Theta)$ as a C^*-subalgebra of $B(\ell^2(\mathbb{Z}^d))$, where u_j is identified with $l_\Theta(e_j)$, $j = 1, \dots, d$.

A matrix $\Theta \in \mathcal{T}_d(\mathbb{R})$ is said to be *nondegenerate* if whenever $x \in \mathbb{Z}^d$ satisfies $\exp(2\pi i \langle x, \Theta y \rangle) = 1$ for all $y \in \mathbb{Z}^d$, then $x = 0$. Otherwise Θ is said to be *degenerate*. For the simplicity of $\mathcal{A}_\Theta$, the following is known.

Theorem 2.2.6. ([18, Theorem 1.9], [23, Theorem 3.7]) *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$. Then the d-dimensional noncommutative torus $\mathcal{A}_\Theta$ is simple if and only if Θ is nondegenerate.*

2.3 Canonical actions on noncommutative tori

In this section, we define canonical actions on noncommutative tori and show that canonical actions on 2-dimensional noncommutative tori are $\mathrm{SL}_2(\mathbb{Z})$ actions on rotation algebras.

For an invertible matrix $A \in \mathrm{GL}_d(\mathbb{Z})$, let $\mathrm{U}_A \in U(\ell^2(\mathbb{Z}^d))$ denote the unitary given by

$$(\mathrm{U}_A\, \xi)(x) = \xi(A^{-1}x)$$

for $\xi \in \ell^2(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$. Then

$$\begin{array}{rccc} \mathrm{Ad}\,\mathrm{U}_A : & B(\ell^2(\mathbb{Z}^d)) & \to & B(\ell^2(\mathbb{Z}^d)), \\ & T & \mapsto & \mathrm{U}_A T \mathrm{U}_A^* \end{array}$$

defines an automorphism of $B(\ell^2(\mathbb{Z}^d))$. Any restrictions of $\mathrm{Ad}\,\mathrm{U}_A$ to subalgebras of $B(\ell^2(\mathbb{Z}^d))$ will also be written as $\mathrm{Ad}\,\mathrm{U}_A$.

Lemma 2.3.1. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ and $A \in \mathrm{GL}_d(\mathbb{Z})$. Then $\mathrm{Ad}\,\mathrm{U}_A$ is an isomorphism from $\mathcal{A}_\Theta$ onto $\mathcal{A}_{(A^{-1})^t\Theta A^{-1}}$ which maps $l_\Theta(y)$ to $l_{(A^{-1})^t\Theta A^{-1}}(Ay)$.*

Proof. Suppose $\xi \in \ell^2(\mathbb{Z}^d)$ and $x, y \in \mathbb{Z}^d$. Then we have

$$\begin{aligned} \mathrm{Ad}\,\mathrm{U}_A(l_\Theta(y))(\xi)(x) &= (\mathrm{U}_A\, l_\Theta(y)\, \mathrm{U}_A^*)(\xi)(x) \\ &= (l_\Theta(y)\, \mathrm{U}_A^*)(\xi)(A^{-1}x) \\ &= \omega_\Theta(y, -y + A^{-1}x)(\mathrm{U}_A^*(\xi))(-y + A^{-1}x) \\ &= \omega_\Theta(y, -y + A^{-1}x)\, \xi(-Ay + x) \\ &= \omega_\Theta(y, A^{-1}x)\, \xi(-Ay + x) \\ &= \omega_{(A^{-1})^t\Theta A^{-1}}(Ay, x)\, \xi(-Ay + x) \\ &= (l_{(A^{-1})^t\Theta A^{-1}}(Ay))(\xi)(x). \end{aligned}$$

Above computation shows that $\mathrm{Ad}_A(l_\Theta(y)) = l_{(A^{-1})^t\Theta A^{-1}}(Ay)$ for $y \in \mathbb{Z}^d$. Since $\{l_\Theta(y) : y \in \mathbb{Z}^d\}$ and $\{l_{(A^{-1})^t\Theta A^{-1}}(Ay) : y \in \mathbb{Z}^d\} = \{l_{(A^{-1})^t\Theta A^{-1}}(y) : y \in \mathbb{Z}^d\}$ generate $\mathcal{A}_\Theta$ and $\mathcal{A}_{(A^{-1})^t\Theta A^{-1}}$, respectively, the conclusion follows. □

Remark 2.3.2. Suppose that $A \in \mathrm{GL}_d(\mathbb{Z})$ satisfies $\Theta = (A^{-1})^t\Theta A^{-1}$ for some $\Theta \in \mathcal{T}_d(\mathbb{R})$. Then by Lemma 2.3.1, $\mathrm{Ad}\,\mathrm{U}_A$ defines an automorphism of $\mathcal{A}_\Theta$ which maps $l_\Theta(y)$ to $l_\Theta(Ay)$ for $y \in \mathbb{Z}^d$.

Definition 2.3.3. Let $\Theta \in \mathcal{T}_d(\mathbb{R})$. Let G_Θ denote the set of all $A \in \mathrm{GL}_d(\mathbb{Z})$

which satisfy $\Theta = (A^{-1})^t \Theta A^{-1}$. That is,

$$G_\Theta = \{A \in \mathrm{GL}_d(\mathbb{Z}) : \Theta = (A^{-1})^t \Theta A^{-1}\}.$$

Lemma 2.3.4. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$. Then G_Θ is a subgroup of $\mathrm{GL}_d(\mathbb{Z})$.*

Proof. Suppose $A, B \in G_\Theta$. Then we see that

$$\begin{aligned}
[(AB^{-1})^{-1}]^t \Theta (AB^{-1})^{-1} &= (BA^{-1})^t \Theta BA^{-1} \\
&= (A^{-1})^t B^t \Theta BA^{-1} \\
&= (A^{-1})^t \Theta A^{-1} \\
&= \Theta.
\end{aligned}$$

Therefore we have $AB^{-1} \in G_\Theta$. This shows that G_Θ is a subgroup of $\mathrm{GL}_d(\mathbb{Z})$. □

Theorem 2.3.5. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$. Define a map $\alpha : G_\Theta \to \mathrm{Aut}(\mathcal{A}_\Theta)$ by $\alpha_A = \mathrm{Ad}\,\mathrm{U}_A$ for $A \in G_\Theta$. Then α is a group action of G_Θ on $\mathcal{A}_\Theta$. Moreover if $\Theta = (\theta_{kj}) \in \mathcal{T}_2(\mathbb{R})$ with $\theta_{12} \neq 0$, then $G_\Theta = \mathrm{SL}_2(\mathbb{Z})$ and α is the $\mathrm{SL}_2(\mathbb{Z})$ action on the rotation algebra $\mathcal{A}_{\theta_{12}}$.*

Proof. Note that each $A \in G_\Theta$ satisfies $\Theta = (A^{-1})^t \Theta A^{-1}$ hence $\alpha_A = \mathrm{Ad}\, U_A$ is an automorphism of $\mathcal{A}_\Theta$ by Remark 2.3.2. To show α is an action, it is enough to show that $\alpha_{AB}(l_\Theta(y)) = \alpha_A \circ \alpha_B(l_\Theta(y))$ for each $y \in \mathbb{Z}^d$. By Remark 2.3.2, we see that

$$\begin{aligned}
\alpha_{AB}(l_\Theta(y)) &= \mathrm{Ad}\, U_{AB}(l_\Theta(y)) \\
&= l_\Theta(ABy) \\
&= \mathrm{Ad}\, U_A \circ \mathrm{Ad}\, U_B(l_\Theta(y)) \\
&= \alpha_A \circ \alpha_B(l_\Theta(y)).
\end{aligned}$$

Therefore α is a group action of G_Θ on $\mathcal{A}_\Theta$. For the other assertion, suppose $A \in \mathrm{GL}_2(\mathbb{Z})$ is given with $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $A \in \mathrm{GL}_2(\mathbb{Z})$, we have $\det(A^{-1}) = (\det A)^{-1} = \pm 1$. Therefore given $\Theta = (\theta_{jk}) \in \mathcal{T}_2(\mathbb{R})$ with $\theta_{12} \neq 0$, we have

$$\begin{aligned}
(A^{-1})^t \Theta A^{-1} &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & \theta_{12} \\ -\theta_{12} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} 0 & (ad-bc)\theta_{12} \\ -(ad-bc)\theta_{12} & 0 \end{pmatrix} \\
&= \det(A^{-1})\Theta.
\end{aligned}$$

Since $\theta_{12} \neq 0$, $(A^{-1})^t \Theta A^{-1} = \Theta$ holds if and only if $\det(A^{-1}) = \det(A) = 1$. This shows that $G_\Theta = \mathrm{SL}_2(\mathbb{Z})$ for $\Theta = (\theta_{jk}) \in \mathcal{T}_2(\mathbb{R})$ with $\theta_{12} \neq 0$. Also for $A \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$\begin{aligned}
\alpha_A(l_\Theta(e_1)) &= l_\Theta(Ae_1) = l_\Theta(ae_1 + ce_2), \\
\alpha_A(l_\Theta(e_2)) &= l_\Theta(Ae_2) = l_\Theta(be_1 + de_2).
\end{aligned}$$

By Lemma 2.2.2(ii), it is easy to see that

$$\begin{aligned}
l_\Theta(ae_1 + ce_2) &= \exp(\pi i a c \theta_{12}) l_\Theta(e_1) l_\Theta(e_2), \\
l_\Theta(be_1 + de_2) &= \exp(\pi i b d \theta_{12}) l_\Theta(e_1) l_\Theta(e_2).
\end{aligned}$$

Comparing this computation with the $\mathrm{SL}_2(\mathbb{Z})$ action in (1.1), under the identification $u_j = l_\Theta(e_j)$, proves the assertion. $\square$

Remarks 2.3.6.

(i) Note that each group G_Θ for $\Theta \in \mathcal{T}_d(\mathbb{R})$ contains the matrices $\pm I_d$. Since $\alpha_{-I_d}(l_\Theta(e_j)) = l_\Theta(e_j)^*$, the restriction of α to the subgroup $\mathbb{Z}_2 =$

$\{\pm I_d\}$ is the flip action (See (1.3)).

(ii) The group G_Θ which acts on the d-dimensional noncommutative torus $\mathcal{A}_\Theta$ by the action α in Theorem 2.3.5 heavily depends on $\Theta \in \mathcal{T}_d(\mathbb{R})$. For example, in case of $d = 2$, we see that $G_\Theta = \mathrm{SL}_2(\mathbb{Z})$ for $\Theta \neq 0$ and $G_\Theta = \mathrm{GL}_2(\mathbb{Z})$ for $\Theta = 0$. If $d > 2$, it is generally hard to exhibit the group G_Θ explicitly so it will be a very difficult question to ask how the group G_Θ varies with Θ. Since the entries of $A \in G_\Theta$ forms an integral solution of a system of equations with coefficients from $\{\theta_{jk}\}$ which may rarely happen if the degree of linearly independence of the coefficients over $\mathbb{Q}$ is large enough, one can roughly expect that G_Θ can vary from $\{\pm I_d\}$ all the way up to $\mathrm{GL}_d(\mathbb{Z})$ according to the degree of linearly independence of the coefficients.

Definition 2.3.7. Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ and G be a subgroup of G_Θ. The restriction of the action $\alpha : G_\Theta \to \mathrm{Aut}(\mathcal{A}_\Theta)$ obtained in Theorem 2.3.5 to G is called the *canonical action* of G on $\mathcal{A}_\Theta$.

Two group actions $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ and $\beta : H \to \mathrm{Aut}(\mathcal{B})$ on C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are said to be *conjugate* if there exist a group isomorphism $\psi : G \to H$ and a C^*-isomorphism $\rho : \mathcal{A} \to \mathcal{B}$ such that $\beta_{\psi(g)} \circ \rho = \rho \circ \alpha_g$ for all $g \in G$. This is an equivalence relation and two conjugate dynamical systems give rise to isomorphic crossed products. In the following proposition we provide a sufficient condition on two matrices $\Theta, \Theta' \in \mathcal{T}_d(\mathbb{R})$ that every canonical action on $\mathcal{A}_\Theta$ is conjugate to a canonical action on $\mathcal{A}_{\Theta'}$.

Proposition 2.3.8. *Let Θ and Θ' be two matrices in $\mathcal{T}_d(\mathbb{R})$ such that*

$$\Theta = (B^{-1})^t \Theta' B^{-1}$$

for some $B \in \mathrm{GL}_d(\mathbb{Z})$. Then we have the following:

(i) *The map* $\psi : G_{\Theta'} \to G_\Theta$, $\psi(A) = BAB^{-1}$, *is a group isomorphism.*

(ii) *If* $\alpha : G' \to \mathrm{Aut}(\mathcal{A}_{\Theta'})$ *is a canonical action of a subgroup* G' *of* $G_{\Theta'}$, *it is conjugate to the canonical action* $\beta : \psi(G') \to \mathrm{Aut}(\mathcal{A}_\Theta)$ *of* $\psi(G')$.

Proof. (i) Let $A \in G_{\Theta'}$. Then $\psi(A) = BAB^{-1} \in G_\Theta$ follows from

$$\begin{aligned}((BAB^{-1})^{-1})^t\Theta(BAB^{-1})^{-1} &= (B^{-1})^t(A^{-1})^tB^t\Theta BA^{-1}B^{-1}\\ &= (B^{-1})^t(A^{-1})^t\Theta' A^{-1}B^{-1}\\ &= (B^{-1})^t\Theta' B^{-1}\\ &= \Theta.\end{aligned}$$

Clearly ψ is a group homomorphism with the inverse $A \mapsto B^{-1}AB$, $A \in G_\Theta$.

(ii) One can check that $\rho := \mathrm{Ad}\, U_B : \mathcal{A}_{\Theta'} \to \mathcal{A}_\Theta$ is an isomorphism such that $\rho(l_{\Theta'}(x)) = l_\Theta(Bx)$ for $x \in \mathbb{Z}^d$, and then for $A \in G'$,

$$\rho \circ \alpha_A(l_{\Theta'}(x)) = \rho(l_{\Theta'}(Ax)) = l_\Theta(BAx).$$

On the other hand, since β is a canonical action, one also has

$$\beta_{\psi(A)} \circ \rho(l_{\Theta'}(x)) = \beta_{\psi(A)}(l_\Theta(Bx)) = l_\Theta(\psi(A)Bx) = l_\Theta(BAx)$$

for all $A \in G'$ and $x \in \mathbb{Z}^d$, which completes the proof. $\square$

Chapter 3

Canonical actions of finite cyclic groups

In this chapter, we study canonical actions on noncommutative tori $\mathcal{A}_\Theta$ by finite cyclic groups $\mathbb{Z}_n$. Since such an action is the restriction of the canonical action $\alpha : G_\Theta \to \mathrm{Aut}(\mathcal{A}_\Theta)$ to a finite cyclic subgroup $\mathbb{Z}_n$ of G_Θ, we need to know whether the group G_Θ contains elements of finite order whenever Θ is given. But this seems very hard because we have to solve the system of equations explained in Remark 2.3.6(ii) under the condition that the solution should form an integral matrix of determinant ± 1. To avoid this difficulty, we plan a strategy for obtaining canonical actions by finite cyclic groups which consists of the following two steps:

- First, we find an integral matrix $A \in \mathrm{GL}_d(\mathbb{Z})$ of finite order n.
- Second, we determine which $\Theta \in \mathcal{T}_d(\mathbb{R})$ admits a canonical action on $\mathcal{A}_\Theta$ by the subgroup $\mathbb{Z}_n = \langle A \rangle$ of G_Θ generated by A obtained in the first step.

In the first section we recall from linear algebra the method for obtaining integral matrices of finite order which are realized as the companion matrices of the cyclotomic polynomials. The second section is devoted to explain explicitly how we determine all the Θ's which admit canonical actions on $\mathcal{A}_\Theta$ by the finite subgroups generated by given matrices of finite order. In the last section, we show that there exist no canonical actions on 3-dimensional noncommutative simple tori by finite cyclic groups except the flip action.

3.1 Companion matrices of cyclotomic polynomials

For a monic polynomial $p(x) = a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + x^d$ of degree d, the *companion matrix* $C_{p(x)}$ associated with p is the $d \times d$ matrix given by

$$C_{p(x)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_{d-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix}. \tag{3.1}$$

It is well known from linear algebra that both of the minimal polynomial and the characteristic polynomial of $C_{p(x)}$ are equal to $p(x)$ itself. Let $\phi(n)$ denote the value of $n \in \mathbb{N}$ under the Euler's totient function ϕ, that is, the number of all primitive nth roots of unity. The nth *cyclotomic polynomial* $\Phi_n(x)$ is the polynomial defined by

$$\Phi_n(x) = \prod_{\gamma} (x - \gamma)$$

where γ ranges over all the distinct primitive nth roots of unity. So $\Phi_n(x)$

can be also written as

$$\Phi_n(x) = \prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}} (x - \exp(2\pi i\,\frac{k}{n})). \tag{3.2}$$

The following properties of the cyclotomic polynomials are well known.

(i) All coefficients of $\Phi_n(x)$ are integers for each $n \in \mathbb{N}$.

(ii) The degree of $\Phi_n(x)$ is $\phi(n)$.

(iii) Every $\Phi_n(x)$ is irreducible over $\mathbb{Q}$.

(iv) The irreducible factorization of $x^n - 1$ is given by $x^n - 1 = \prod_{k|n} \Phi_k(x)$.

Notation 3.1.1. For $n \in \mathbb{N}$, let C_n denote the companion matrix of the nth cyclotomic polynomial $\Phi_n(x)$.

Lemma 3.1.2. *If $n \ge 3$, then we have $\sum_{\substack{1\le k\le n\\ \gcd(k,n)=1}} k = \frac{\phi(n)}{2}n$.*

Proof. Suppose that $1 \le k \le n$ and $\gcd(k,n) = 1$. Then there exist $l, m \in \mathbb{Z}$ satisfying $lk + mn = 1$. Recall that the existence of such $l, m \in \mathbb{Z}$ is equivalent to $\gcd(k,n) = 1$. Since $(-l)(n-k) + (m+l)n = lk + mn = 1$, we see that $1 \le n-k \le n$ and $\gcd(n-k,n) = 1$. Note that $k = n-k$ (or $2k = n$) holds if and only if $k = 1$ and $n = 2$. Therefore if $n \ge 3$, $\phi(n)$ is even and we have

$$\begin{aligned}\sum_{\substack{1\le k\le n\\ \gcd(k,n)=1}} k &= \sum_{\substack{1\le k\le [n/2]\\ \gcd(k,n)=1}} k + (n-k)\\ &= \frac{\phi(n)}{2}n.\end{aligned}$$

□

Proposition 3.1.3. *Let $n \in \mathbb{N}$ and $d = \phi(n)$. Then we have the following.*

(i) *C_n is an element in $\mathrm{GL}_d(\mathbb{Z})$ of finite order n.*

(ii) *If $n \geq 3$, then d is even.*

(iii) *C_n is diagonalizable in $\mathbb{C}$. More precisely, there exists an invertible matrix $U \in \mathbb{M}_d(\mathbb{C})$ satisfying $UC_nU^{-1} = \mathrm{diag}(\gamma_1, \ldots, \gamma_d)$ where γ_k's are the distinct primitive nth roots of unity.*

(iv) *If $n \geq 3$, then $C_n \in \mathrm{SL}_d(\mathbb{Z})$.*

Proof. (i) Since all the coefficients of $\Phi_n(x)$ are integers and since the degree of $\Phi_n(x)$ is equal to d, the companion matrix C_n of $\Phi_n(x)$ is a $d \times d$ integral matrix. To show C_n is a matrix of order n, first note that $\Phi_n(x)$ divides $x^n - 1$. By Cayley-Hamilton theorem, we have $C_n^n = I_d$. So C_n has a finite order k which divides n. Since the minimal polynomial of C_n is $\Phi_n(x)$, $\Phi_n(x)$ must divide $x^k - 1$. But $\Phi_n(x)$ can be one of the irreducible factors of $x^k - 1$ if and only if n divides k, we conclude that $k = n$.

(ii) It is straight forward from the proof of Lemma 3.1.2.

(iii) Since the characteristic polynomial $\Phi_n(x)$ of C_n is of degree d and has d distinct roots consisting of the primitive nth roots of unity, the conclusion follows.

(iv) By (iii) we know that $\det C_n$ is equal to the product of all the distinct primitive nth roots of unity. Note that the set of all primitive nth roots of unity consists of the scalars of the form $\exp(2\pi i \frac{k}{n})$ where k is an any integer such that $1 \leq k \leq n$ and $\gcd(\mathrm{k}, \mathrm{n}) = 1$. Therefore

$$\begin{aligned}\det C_n &= \prod_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} \exp(2\pi i \frac{k}{n}) \\ &= \exp(\frac{2\pi i}{n} \sum_{\substack{1 \leq k \leq n \\ \gcd(k,n)=1}} k) \\ &= \exp(d\pi i),\end{aligned}$$

where the third equality comes from Lemma 3.1.2. Since d is even for $n \geq 3$ by (ii), we have $\det C_n = 1$ for $n \geq 3$. □

Remark 3.1.4. In this thesis, $d = \phi(n)$ will be considered as the dimension of noncommutative tori on which the cyclic group $\mathbb{Z}_n = \langle C_n \rangle$ generated by C_n acts canonically. Therefore we are mainly concerned with those n satisfying $d = \phi(n) \geq 2$ which happen exactly when $n \geq 3$, in this case, $d = \phi(n)$ is even and $C_n \in \mathrm{SL}_d(\mathbb{Z})$.

3.2 Noncommutative tori admitting canonical actions by finite cyclic groups

Recall that for a preassigned $\Theta \in \mathcal{T}_d(\mathbb{R})$,

$$G_\Theta = \{A \in \mathrm{GL}_d(\mathbb{Z}) : \Theta = (A^{-1})^t \Theta A^{-1}\}$$

forms a group. Thus $\Theta = (A^{-1})^t \Theta A^{-1}$ if and only if $\Theta = A^t \Theta A$ for $A \in \mathrm{GL}_d(\mathbb{Z})$. In this section, given $A \in \mathrm{GL}_d(\mathbb{Z})$ we want to determine all $\Theta \in \mathcal{T}_d(\mathbb{R})$ such that $A \in G_\Theta$. We will use the following notation for the set of those Θ's.

Notation 3.2.1. For $A \in \mathrm{GL}_d(\mathbb{Z})$, let $\mathcal{T}_{d,A}(\mathbb{R})$ be the subset of $\mathcal{T}_d(\mathbb{R})$ given by

$$\mathcal{T}_{d,A}(\mathbb{R}) = \{\Theta \in \mathcal{T}_d(\mathbb{R}) : \Theta = A^t \Theta A\}.$$

Once $A \in \mathrm{GL}_d(\mathbb{Z})$ is given, the task of determining which Θ belongs to $\mathcal{T}_{d,A}(\mathbb{R})$ is nothing but solving a system of linear equations. For each $\Theta \in \mathcal{T}_{d,A}(\mathbb{R})$, the subgroup $\langle A \rangle$ of G_Θ generated by A admits a canonical action on the noncommutative torus $\mathcal{A}_\Theta$. By taking $\Theta \in \mathcal{T}_{d,A}(\mathbb{R})$ whenever a matrix A of finite order is given, one can obtain plentiful examples of canonical actions on noncommutative tori by finite cyclic groups. Note that the companion matrices of cyclotomic polynomials can provide us elements of finite order in $\mathrm{GL}_d(\mathbb{Z})$ for even d. Since we know all the examples of canonical actions by finite cyclic subgroups on the rotation algebras which is the case of $d = 2$, next possible examples which come to our mind will be from the case of $d = 4$.

It is easy to see that the set of $n \in \mathbb{N}$ satisfying $\phi(n) = 4$ is $\{5, 8, 10, 12\}$. For such n, the nth cyclotomic polynomials $\Phi_n(x)$'s are computed as below.

$$\begin{aligned}
\Phi_5(x) &= 1 + x + x^2 + x^3 + x^4, \\
\Phi_8(x) &= 1 + x^4, \\
\Phi_{10}(x) &= 1 - x + x^2 - x^3 + x^4, \\
\Phi_{12}(x) &= 1 - x^2 + x^4
\end{aligned}$$

And the corresponding companion matrices C_n's are the following:

$$\begin{aligned}
C_5 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad C_8 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
C_{10} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad C_{12} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{3.3}$$

Note that each C_n is of order n by Proposition 3.1.3(i) and $C_n \in \mathrm{SL}_4(\mathbb{Z})$ by Proposition 3.1.3(iv) for $n = 5, 8, 10, 12$. By definition, $\mathcal{T}_{4,C_n}(\mathbb{R})$ is the set of all $\Theta \in \mathcal{T}_4(\mathbb{R})$ with $\Theta - C_n^t \Theta C_n = 0$ for $n = 5, 8, 10, 12$. To find elements of $\mathcal{T}_{4,C_n}(\mathbb{R})$, let $\Theta = (\theta_{kj}) \in \mathcal{T}_4(\mathbb{R})$. Then a simple calculation shows that

$$\Theta - C_5^t \Theta C_5 = \begin{pmatrix} 0 & \theta_{12} - \theta_{23} & \theta_{13} - \theta_{24} & -\theta_{12} + \theta_{14} + \theta_{23} + \theta_{24} \\ * & 0 & \theta_{23} - \theta_{34} & -\theta_{13} - \theta_{23} + \theta_{24} + \theta_{34} \\ * & * & 0 & -\theta_{14} - \theta_{24} \\ * & * & * & 0 \end{pmatrix},$$

$$\Theta - C_8^t \Theta C_8 = \begin{pmatrix} 0 & \theta_{12} - \theta_{23} & \theta_{13} - \theta_{24} & -\theta_{12} + \theta_{14} \\ * & 0 & \theta_{23} - \theta_{34} & -\theta_{13} + \theta_{24} \\ * & * & 0 & -\theta_{14} + \theta_{34} \\ * & * & * & 0 \end{pmatrix},$$

$$\Theta - C_{10}^t \Theta C_{10} = \begin{pmatrix} 0 & \theta_{12} - \theta_{23} & \theta_{13} - \theta_{24} & -\theta_{12} + \theta_{14} + \theta_{23} - \theta_{24} \\ * & 0 & \theta_{23} - \theta_{34} & -\theta_{13} + \theta_{23} + \theta_{24} - \theta_{34} \\ * & * & 0 & -\theta_{14} + \theta_{24} \\ * & * & * & 0 \end{pmatrix},$$

$$\Theta - C_{12}^t \Theta C_{12} = \begin{pmatrix} 0 & \theta_{12} - \theta_{23} & \theta_{13} - \theta_{24} & -\theta_{12} + \theta_{14} - \theta_{23} \\ * & 0 & \theta_{23} - \theta_{34} & -\theta_{13} + \theta_{24} \\ * & * & 0 & -\theta_{14} + 2\theta_{34} \\ * & * & * & 0 \end{pmatrix}.$$

Thus $\mathcal{T}_{4,C_n}(\mathbb{R})$ can be described with two real parameters μ, θ as follows.

$$\mathcal{T}_{4,C_5}(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \theta & \mu & -\mu \\ -\theta & 0 & \theta & \mu \\ -\mu & -\theta & 0 & \theta \\ \mu & -\mu & -\theta & 0 \end{pmatrix} : \theta, \mu \in \mathbb{R} \right\}, \tag{3.4}$$

$$\mathcal{T}_{4,C_8}(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \mu & \theta & \mu \\ -\mu & 0 & \mu & \theta \\ -\theta & -\mu & 0 & \mu \\ -\mu & -\theta & -\mu & 0 \end{pmatrix} : \theta, \mu \in \mathbb{R} \right\}, \tag{3.5}$$

$$\mathcal{T}_{4,C_{10}}(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \theta & \mu & \mu \\ -\theta & 0 & \theta & \mu \\ -\mu & -\theta & 0 & \theta \\ -\mu & -\mu & -\theta & 0 \end{pmatrix} : \theta, \mu \in \mathbb{R} \right\}, \tag{3.6}$$

$$\mathcal{T}_{4,C_{12}}(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \mu & \theta & 2\mu \\ -\mu & 0 & \mu & \theta \\ -\theta & -\mu & 0 & \mu \\ -2\mu & -\theta & -\mu & 0 \end{pmatrix} : \theta, \mu \in \mathbb{R} \right\}. \tag{3.7}$$

From the above list, we see that there are many 4-dimensional noncommutative tori which admit the canonical action by the finite group $\mathbb{Z}_n$ for $n = 5, 8, 10, 12$:

Example 3.2.2. *Let $\Theta \in \mathcal{T}_4(\mathbb{R})$ be a real skew symmetric 4×4 matrix and $n = 5, 8, 10,$ or 12. If $\Theta \in \mathcal{T}_{4,C_n}(\mathbb{R})$, the noncommutative 4-torus $\mathcal{A}_\Theta$ admits the canonical action by the finite group $\mathbb{Z}_n$ generated by C_n in* (3.3).

Remark 3.2.3. Even though our method for obtaining canonical actions on 4-dimensional noncommutative tori by finite cyclic groups can be definitely applied to obtain actions on higher dimensional noncommutative tori, this seems to be very restrictive since in this way we can obtain actions only on even dimensional tori and also we certainly do not obtain all the canonical actions by finite cyclic groups of all possible orders on torus of given dimension. For example, there are matrices of finite order $2, 3, 4, 6$ in $\mathrm{GL}_4(\mathbb{Z})$ which we do not deal with in the above example. But these flaws can be overcome by taking a matrix $A \in \mathrm{GL}_N(\mathbb{Z})$ into account which are obtained from letting C_n's and ± 1's sit inside on the diagonal of A so that the sum of the number of ± 1's and $\phi(n)$'s is equal to N. One may refer to [13] for the further explanation. Applying this modification, one can finally obtain canonical actions on arbitrary dimensional noncommutative tori by finite cyclic groups of all possible finite orders.

3.3 Canonical actions on 3-dimensional simple tori by finite cyclic groups

As in the 2-dimensional case, if we have the complete list of group elements of finite order in $\mathrm{GL}_d(\mathbb{Z})$ up to conjugacy, then by Proposition 2.3.8 we would be able to find all canonical actions by finite cyclic groups on d-dimensional tori. Actually this is possible for $d = 3$ by virtue of the list (see Table 3.1) established in [24].

Table 3.1: Elements of finite order in $\mathrm{GL}_3(\mathbb{Z})$

Order	Generators
2	$A_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ A_2^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$ $A_3^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ A_4^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ & -1 & 0 \end{pmatrix},$ $A_5^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
3	$A_1^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix},\ A_2^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
4	$A_1^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ A_2^4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$ $A_3^4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ A_4^4 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Order	Generators
6	$A_1^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, $A_2^6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$, $A_3^6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, $A_4^6 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

Theorem 3.3.1. *The only canonical action by a nontrivial finite cyclic group on a simple 3-dimensional torus is the flip action; if $A \in \mathrm{GL}_3(\mathbb{Z})$, $A \neq -I_3$, is a matrix in Table 3.1, every $\Theta \in \mathcal{T}_{3,A}(\mathbb{R})$ is degenerate.*

Proof. If a 3-torus $\mathcal{A}_{\Theta'}$ admits a canonical action of a finite cyclic group $G' \subset G_{\Theta'}$, then G' must be conjugate to a cyclic group generated by a matrix A in the table, so that there exists $B \in \mathrm{GL}_3(\mathbb{Z})$ such that $G' = \langle B^{-1}AB \rangle$ is generated by $B^{-1}AB$. But then by Proposition 2.3.8, the cyclic group $\langle A \rangle$ canonically acts on the 3-torus $\mathcal{A}_\Theta$, where $\Theta := (B^{-1})^t\Theta' B^{-1}$. Also, it is rather obvious that Θ is nondegenerate exactly when Θ' is nondegenerate. Therefore by Theorem 2.2.6 it is enough to show that if A is one of the matrices listed in the table and $A \neq -I_3 (= A_5^2)$, every $\Theta \in \mathcal{T}_{3,A}(\mathbb{R})$ should be degenerate.

Recall that $\Theta \in \mathcal{T}_d(\mathbb{R})$ is degenerate if there exists a nonzero $x \in \mathbb{Z}^d$ such that $\exp(2\pi i \langle \Theta x, y \rangle) = 1$ for all $y \in \mathbb{Z}^d$, or equivalently if there is a nonzero $x \in \mathbb{Z}^d$ with $\langle \Theta x, e_j \rangle \in \mathbb{Z}$ for all $j = 1, \cdots, d$. Thus, to obtain the degeneracy of $\Theta \in \mathcal{T}_{3,A}(\mathbb{R})$, we find nonzero elements $x \in \mathbb{Z}^3$ with $\Theta x \in \mathbb{Z}^3$.

It is rather tedious to do the same calculation with all the matrices in the table, so here we only do with $A = A_1^2$ and leave the rest to readers. If $\Theta = (\theta_{kj}) \in \mathcal{T}_{3,A}(\mathbb{R})$, that is $\Theta - A^t\Theta A = \begin{pmatrix} 0 & 2\theta_{12} & 2\theta_{13} \\ -2\theta_{12} & 0 & 0 \\ -2\theta_{13} & 0 & 0 \end{pmatrix}$ is the zero matrix, then Θ must be of the form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & -s & 0 \end{pmatrix}$ for an $s \in \mathbb{R}$. Any such matrix Θ is degenerate; in fact, $\Theta x = (0,0,0)^t \in \mathbb{Z}^3$ for any $x = (k,0,0)^t \in \mathbb{Z}^3$. $\square$

Chapter 4

Crossed products of noncommutative simple tori by canonical actions of finite groups

In this chapter, we study the crossed products of noncommutative tori by canonical actions of finite groups under the standing assumptions:

- $\Theta \in \mathcal{T}_d(\mathbb{R})$ is nondegenerate, that is, $\mathcal{A}_\Theta$ is simple.
- G is a finite subgroup of G_Θ.
- $\alpha : G \to \mathrm{Aut}(\mathcal{A}_\Theta)$ is the canonical action of G on $\mathcal{A}_\Theta$.

In the first section, following the steps conducted in [4] and employing various results therein, we show that the crossed products of higher dimensional noncommutative simple tori by the actions of finite groups also belong to the class of C^*-algebras with tracial rank zero satisfying the Universal Coefficient Theorem(UCT). Also we recall the classification theorem [16, Theorem 5.2] for algebras in such class due to H. Lin and we employ one of its variation [18, Proposition 3.7] by N. C. Phillips to our situation to conclude that by computing K-theory we can determine the crossed products of simple tori by canonical actions of finite groups. In the second section, we mainly focus on the canonical actions on simple tori by finite cyclic groups generated by companion matrices of cyclotomic polynomials and study K-theory of the crossed products of them.

4.1 Structure of the crossed products of simple tori by canonical actions of finite groups

4.1.1 Tracial rank of crossed products

We begin this section by stating two known definitions.

Definition 4.1.1. [19, Definition 1.2] Let $\mathcal{A}$ be an infinite dimensional simple separable unital C^*-algebra, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset \mathcal{A}$, every $\epsilon > 0$, and every positive element $x \in \mathcal{A}$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in \mathcal{A}$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \epsilon$ for all $g, h \in G$,

(2) $\|e_g a - a e_g\| < \epsilon$ for all $g \in G$ and $a \in F$,

(3) with $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of $\mathcal{A}$ generated by x,

(4) with e in (3), we have $\|exe\| > 1 - \epsilon$.

Definition 4.1.2. [15, Definition 3.1], [19, Proposition 2.3] Let $\mathcal{A}$ be a simple unital C^*-algebra. Then $\mathcal{A}$ is said to have *tracial rank zero* if for every finite subset $F \subset \mathcal{A}$, every $\epsilon > 0$, and every nonzero positive element $x \in \mathcal{A}$, there exists a projection $p \in \mathcal{A}$ and a unital finite dimensional subalgebra $D \subset p\mathcal{A}p$ such that:

(1) $\|[a, p]\| < \epsilon$ for all $a \in F$,

(2) $\mathrm{dist}(pap, D) < \epsilon$ for all $a \in F$,

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{x\mathcal{A}x}$.

In this case, we write $TR(\mathcal{A}) = 0$.

A finite group action which has the tracial Rokhlin property is known to enjoy the following permanence property due to N. C. Phillips [19].

Theorem 4.1.3. [19, Corollary 1.6, Theorem 2.6] *Let $\mathcal{A}$ be an infinite dimensional separable unital simple C^*-algebra and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be a finite group action which has the tracial Rokhlin property. Then the crossed product $\mathcal{A} \rtimes_\alpha G$ is simple. Moreover, if $\mathcal{A}$ has tracial rank zero, then the crossed product $\mathcal{A} \rtimes_\alpha G$ also has tracial rank zero.*

Now we are ready to state the following Theorem obtained by combining various known results.

Theorem 4.1.4. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ be nondegenerate and let G be a finite subgroup of G_Θ and $\alpha : G \to \mathrm{Aut}(\mathcal{A}_\Theta)$ be the canonical action of G on $\mathcal{A}_\Theta$. Then the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is a simple C^*-algebra which has tracial rank zero.*

Proof. Let $A = (a_{kj}) \in G \setminus \{I_d\}$. Then by the identification of $l_\Theta(e_j)$ with u_j as in Proposition 2.2.3, we see that $\alpha_A(u_j) = \rho u_1^{a_{1j}} \ldots u_d^{a_{dj}}$ for some $\rho \in \mathbb{T}$. Since $A \neq I_d$, there exists $j \in \{1, \ldots, d\}$ satisfying $(a_{1j}, \ldots, a_{dj}) \neq e_j$ where e_j is the jth standard basis element of $\mathbb{Z}^d$. Let τ be the unique tracial state of $\mathcal{A}_\Theta$ [18, Theorem 1.9] and π_τ be the Gelfand-Naimark-Segal representation associated with τ. By [4, Lemma 5.10], $(\alpha_A)''$ is outer where $(\alpha_A)''$ denotes the automorphism of $\pi_\tau(\mathcal{A})$ determined by α_A. Since this holds for every $A \in G \setminus \{I_d\}$, α has the tracial Rokhlin property by [4, Theorem 5.5]. Since $\mathcal{A}_\Theta$ has tracial rank zero [18, Theorem 3.5], the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ also has tracial rank zero by Theorem 4.1.3. □

4.1.2 The Universal Coefficient Theorem for crossed products

Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ and G be a subgroup of G_Θ. The group G acts on $\mathbb{Z}^d$ via matrix multiplication

$$(A, x) \mapsto Ax : G \times \mathbb{Z}^d \to \mathbb{Z}^d \tag{4.1}$$

and we have the semidirect product group $\mathbb{Z}^d \rtimes G$ with the group multiplication

$$(x, A)(y, B) = (x + Ay, AB) \tag{4.2}$$

for $x, y \in \mathbb{Z}^d$ and $A, B \in G$. Note that the cocycle ω_Θ is *invariant* under the above action, namely $\omega_\Theta(Ax, Ay) = \omega_\Theta(x, y)$ for $A \in G$ and $x, y \in \mathbb{Z}^d$, which is obvious from

$$\begin{aligned}\exp(\pi i\langle \Theta Ax, Ay\rangle) &= \exp(\pi i\langle A^t\Theta Ax, y\rangle)\\ &= \exp(\pi i\langle \Theta x, y\rangle).\end{aligned}$$

In this case, we can easily check that the map $\widetilde{\omega}_\Theta$ defined by

$$\widetilde{\omega}_\Theta((x, A), (y, B)) = \omega_\Theta(x, Ay) \tag{4.3}$$

is a 2-cocycle of $\mathbb{Z}^d \rtimes G$. The following lemma which is a special case of [4, Lemma 2.1] shows that the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ can be realized as the twisted group algebra of group $\mathbb{Z}^d \rtimes G$ twisted by the 2-cocycle $\widetilde{\omega}_\Theta$.

Lemma 4.1.5. ([4, Lemma 2.1]) *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ and let α be the canonical action on $\mathcal{A}_\Theta$ by a subgroup G of G_Θ. Let $\mathbb{Z}^d \rtimes G$ be the semidirect product with operation given by (4.2) and $\widetilde{\omega}_\Theta$ be the 2-cocycle on $\mathbb{Z}^d \rtimes G$ given by (4.3). Then there is an isomorphism*

$$\Phi : C^*(\mathbb{Z}^d \rtimes G, \widetilde{\omega}_\Theta) \to C^*(\mathbb{Z}^d, \omega_\Theta) \rtimes_\alpha G$$

such that for $f \in \ell^1(\mathbb{Z}^d \rtimes G, \widetilde{\omega}_\Theta)$, the ℓ^1-function $\Phi(f) \in \ell^1(G, \ell^1(\mathbb{Z}^d, \omega_\Theta))$ is given by $\Phi(f)(A) = f(\cdot\,, A)$ for $A \in G$.

Remark 4.1.6. Even if α denotes the canonical action of $G \subset G_\Theta$ on $\mathcal{A}_\Theta$, we will also use the same symbol α to denote the conjugation action of G on $\mathbb{Z}^d$ in the semidirect product. Certainly we do not confuse the meaning of each α when we refer to the isomorphism

$$C^*(\mathbb{Z}^d \rtimes_\alpha G, \widetilde{\omega}_\Theta) \cong \mathcal{A}_\Theta \rtimes_\alpha G.$$

A C^*-algebra $\mathcal{A}$ is said to *satisfy the Universal Coefficient Theorem*(UCT) if $\mathcal{A}$ is KK-equivalent to an abelian C^*-algebra. The UCT is one of the key regularity conditions imposed on most of the classification theorems for nuclear C^*-algebras.

Theorem 4.1.7. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ be nondegenerate and G be a finite subgroup of G_Θ. If $\alpha : G \to \operatorname{Aut}(\mathcal{A}_\Theta)$ is the canonical action of G on A_Θ then the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ satisfies the Universal Coefficient Theorem.*

Proof. The 2-cocycle ω_Θ given in (2.4) is invariant under the action of G on $\mathbb{Z}^d$. By Lemma 4.1.5, the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is isomorphic to the twisted group algebra $C^*(\mathbb{Z}^d \rtimes G, \widetilde{\omega}_\Theta)$. Note that $\mathbb{Z}^d \rtimes G$ is amenable and a closed subgroup of $\mathbb{R}^d \rtimes G$ which is almost connected. Therefore $\mathcal{A}_\Theta \rtimes_\alpha G$ satisfies the Universal Coefficient Theorem by [4, Corollary 6.2]. $\square$

4.1.3 Classification theorem for crossed products

Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ be nondegenerate and G be a finite subgroup of G_Θ. Let $\alpha : G \to \mathrm{Aut}(\mathcal{A}_\Theta)$ be the canonical action of G on $\mathcal{A}_\Theta$. By Theorem 4.1.4 and Theorem 4.1.7, the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ belongs to the class of unital separable simple C^*-algebras with tracial rank zero which satisfy the Universal Coefficient Theorem. For such a class of C^*-algebras there is a very useful classification theorem due to Huaxin Lin which we state below.

Theorem 4.1.8. ([16, Theorem 5.2]) *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital separable simple nuclear C^*-algebras with tracial topological rank zero which satisfy the Universal Coefficient Theorem. Then $\mathcal{A} \cong \mathcal{B}$ if and only if they have isomorphic Elliott invariants, that is,*

$$(K_0(\mathcal{A}), K_0(\mathcal{A})_+, [1_\mathcal{A}], K_1(\mathcal{A})) \cong (K_0(\mathcal{B}), K_0(\mathcal{B})_+, [1_\mathcal{B}], K_1(\mathcal{B})).$$

Since simple unital AF algebras satisfy all the conditions of the above theorem, if the Elliott invariant of $\mathcal{A}_\Theta \rtimes_\alpha G$ is isomorphic to that of such an AF algebra, one can conclude that the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is an AF algebra, which was successfully done in [4] for $\mathcal{A}_\theta \rtimes_\alpha F$ with all $\theta \in \mathbb{R}\backslash\mathbb{Q}$ and all finite subgroups F of $\mathrm{SL}_2(\mathbb{Z})$. The following proposition by N.C. Phillips also says that to see whether the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is AF, we only need to know its K-groups:

Proposition 4.1.9. ([18, Proposition 3.7]) *Let $\mathcal{A}$ be a simple infinite dimensional separable unital nuclear C^*-algebra with tracial rank zero and which* *satisfies the Universal Coefficient Theorem. Then $\mathcal{A}$ is a simple AH algebra with real rank zero and no dimension growth. If $K_*(\mathcal{A})$ is torsion free, $\mathcal{A}$ is an AT algebra. If, in addition, $K_1(\mathcal{A}) = 0$, then $\mathcal{A}$ is an AF algebra.*

Remark 4.1.10. We can summarize what the classification results above imply in our setting as follows: If Θ is a nondegenerate skew symmetric real $d \times d$ matrix and $\alpha : G \to \mathrm{Aut}(\mathcal{A}_\Theta)$ is the canonical action of a finite group $G \subset G_\Theta$, the simple crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is an AF algebra if and only if $K_0(\mathcal{A}_\Theta \rtimes_\alpha G)$ is torsion free and $K_1(\mathcal{A}_\Theta \rtimes_\alpha G) = 0$.

Recall from Lemma 4.1.5 that the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is isomorphic to the twisted group algebra $C^*(\mathbb{Z}^d \rtimes G, \widetilde{\omega}_\Theta)$ if α is the canonical action on a d-dimensional noncommutative torus $\mathcal{A}_\Theta$ by a subgroup G of G_Θ. We close this section by showing that the K-theory of the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is same as the one of the usual (untwisted) group C^*-algebra $C^*(\mathbb{Z}^d \rtimes G)$ if G is finite.

Theorem 4.1.11. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ and G be a finite subgroup of G_Θ. Let $\mathbb{Z}^d \rtimes G$ be the semidirect product with group operation given by (4.2) and $\widetilde{\omega}_\Theta$ be the 2-cocycle on $\mathbb{Z}^d \rtimes G$ given by (4.3). Then we have*

$$K_*(C^*(\mathbb{Z}^d \rtimes G, \widetilde{\omega}_\Theta)) \cong K_*(C^*(\mathbb{Z}^d \rtimes G)).$$

Proof. It is easy to see that the 2-cocycle $\widetilde{\omega}_\Theta$ is homotopic (in the sense of [4, Theorem 0.3]) to the trivial one via

$$\Omega : (\mathbb{Z}^d \rtimes G) \times (\mathbb{Z}^d \rtimes G) \to C([0,1], \mathbb{T})$$

defined by

$$\Omega((x, A), (y, B))(t) := \exp(2\pi i t \langle \Theta x, Ay \rangle)$$

for $x, y \in \mathbb{Z}^d$, $A, B \in G$ and $t \in [0,1]$. Therefore the conclusion follows directly from [4, Theroem 0.3] since the group $\mathbb{Z}^d \rtimes G$ is amenable if G is finite. $\square$

4.2 K-theory of the crossed products

In this section we provide a sufficient condition that the K_1-group of the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is trivial under the assumption that α is the canonical action of the cyclic group $G \cong \mathbb{Z}_n$ generated by the companion matrix C_n on a d-dimensional noncommutative torus $\mathcal{A}_\Theta$ with $d = \phi(n)$. By Theorem 4.1.11, it is enough to know the K-theory of the group C^*-algebra of the form $C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)$ instead of the K-theory of $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$. Recall from Remark 4.1.6 that we use the same symbol α to denote the conjugation action of $\mathbb{Z}_n$ on $\mathbb{Z}^d$.

In [3, 14], the topological K-groups of the group C^*-algebra $C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)$ were provided under the assumption that the conjugation action α of $\mathbb{Z}_n$ on $\mathbb{Z}^d$ is *free outside the origin* $0 \in \mathbb{Z}^d$, that is, $\alpha_k(x) \neq x$ for every non-identity element $k \in \mathbb{Z}_n$ and every non-identity element $x \in \mathbb{Z}^d$.

Theorem 4.2.1. ([14, Theorem 0.1], [3, Theorem 0.3]) *Let $n, d \in \mathbb{N}$. Consider the extension of groups $1 \to \mathbb{Z}^d \to \mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n \to \mathbb{Z}_n \to 1$ such that conjugation action α of $\mathbb{Z}_n$ on $\mathbb{Z}^d$ is free outside the origin. Then*

$$K_0(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \cong \mathbb{Z}^{s_0},$$

$$K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \cong \mathbb{Z}^{s_1},$$

where

$$s_0 = \sum_{l \geq 0} \mathrm{rk}_{\mathbb{Z}}((\Lambda^{2l}\mathbb{Z}^d)^{\mathbb{Z}_n}) + \sum_{(M) \in \mathcal{M}} (|M| - 1) \tag{4.4}$$

and

$$s_1 = \sum_{l \geq 0} \mathrm{rk}_{\mathbb{Z}}((\Lambda^{2l+1}\mathbb{Z}^d)^{\mathbb{Z}_n}). \tag{4.5}$$

In particular, $s_1 = 0$ if n is even. Also, if $n > 2$ is prime and $d = n-1$, then

$$s_1 = \frac{2^{n-1} - (n-1)^2}{2n}. \tag{4.6}$$

Remarks 4.2.2.

(1) The conjugation action α of $\mathbb{Z}_n$ on $\mathbb{Z}^d$ induces an action $\Lambda(\alpha)$ of $\mathbb{Z}_n$ on each lth exterior power $\Lambda^l\mathbb{Z}^d$ of $\mathbb{Z}^d$. Since the integer ring $\mathbb{Z}$ is PID, the fixed point set $(\Lambda^l\mathbb{Z}^d)^{\mathbb{Z}_n}$ is a free $\mathbb{Z}$-submodule of $\Lambda^l\mathbb{Z}^d$ for each integer $l \geq 0$. So the $\mathbb{Z}$-rank or $\mathrm{rk}_{\mathbb{Z}}$ in (4.4) and (4.5) of Theorem 4.2.1 makes sense. $\mathcal{M}$ in (4.4) of Theorem 4.2.1 denotes the set of conjugacy classes of maximal finite subgroups of $\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n$. Whatever the conjugation action α would be, one sees that the K_i-group of $C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)$ is free abelian group of rank s_i for $i = 0, 1$ provided α is free outside the origin. In particular, they are torsion free.

(2) Suppose $n > 2$ is prime and $d = n - 1$. Then $s_1 = 0$ in (4.6) if and only if $2^{n-1} = (n-1)^2$. By the unique prime factorization property of natural numbers, $2^{n-1} = (n-1)^2$ is possible at least if $n - 1 = 2^k$ for some integer $k > 0$ and in this case we have $2^{2^k} = 2^{2k}$ which holds only if $k = 1, 2$. So for prime n with $d = n-1$ we have $s_1 = 0$ if and only if $n = 2, 3, 5$.

4.2.1 Conjugation actions free outside the origin

To be able to apply Theorem 4.2.1 to the case of group C^*-algebra $C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)$, where the conjugation action α of $\mathbb{Z}_n = \langle C_n \rangle$ on $\mathbb{Z}^d$ is given by matrix multiplication as in (4.1), we need to show that α is free outside the origin.

Proposition 4.2.3. *The conjugation action α of $\mathbb{Z}_n = \langle C_n \rangle$ on $\mathbb{Z}^d$ in the semidirect product group $\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n$ is free outside the origin, that is, $C_n^k x \neq x$ for all $k = 1, \ldots, n-1$ and nonzero $x \in \mathbb{Z}^d$.*

Proof. Assuming to the contrary, suppose there exist $k \in \{1, \ldots, n-1\}$ and nonzero $x \in \mathbb{Z}^d$ satisfying $C_n^k x = x$. By Proposition 3.1.3(iii), C_n is diagonalizable over $\mathbb{C}$ and there exists an invertible matrix $U \in \mathbb{M}_d(\mathbb{C})$ such that $C_n = U \operatorname{diag}(\gamma_1, \ldots, \gamma_d) U^{-1}$ where γ_j's are the distinct primitive nth roots of unity. So we have $C_n^k = U \operatorname{diag}(\gamma_1^k, \ldots, \gamma_d^k) U^{-1}$, hence

$$\operatorname{diag}(\gamma_1^k, \ldots, \gamma_d^k) U^{-1} x = U^{-1} x. \tag{4.7}$$

Since U is invertible and x is nonzero, the equation (4.7) implies that the matrix $\operatorname{diag}(\gamma_1^k, \ldots, \gamma_d^k)$ has 1 as its eigenvalue with corresponding eigenvector $U^{-1}x$. Therefore we have $\gamma_j^k = 1$ for some $j \in \{1, \ldots, d\}$. We may write $\gamma_j = \exp(2\pi i \frac{m}{n})$ for some $1 \leq m \leq n$ with $\gcd(\mathrm{m}, \mathrm{n}) = 1$. Since $\exp(2\pi i \frac{mk}{n}) = 1$, n must divide mk. By the condition $\gcd(\mathrm{m}, \mathrm{n}) = 1$, n must divide k which gives a contradiction since $1 \leq k \leq n-1$. $\square$

The above proposition, together with Proposition 4.1.9, Theorem 4.1.11, Remark 4.2.2(ii) and Theorem 4.2.1, gives the following:

Proposition 4.2.4. *Let $d = \phi(n)$ and α be the canonical action of the finite cyclic group $\mathbb{Z}_n = \langle C_n \rangle$ on a noncommutative simple d-torus $\mathcal{A}_\Theta$. Then the crossed product $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is an AT algebra. Moreover it is an AF algebra if and only if $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) = 0$; in particular if n is even, $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is always an AF algebra. If n is prime, $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is AF if and only if $n = 2, 3, 5$.*

Remark 4.2.5. Note that Proposition 4.2.4 above does not guarantee yet the exsistence of non-AF crossed products until one does find canonical actions on simple tori by finite groups $\mathbb{Z}_n = \langle C_n \rangle$.

4.2.2 A sufficient condition for $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \neq 0$

Now let $n \geq 3$ be an odd number with $d = \phi(n)$ and α be the conjugation action of $\mathbb{Z}_n = \langle C_n \rangle$ on $\mathbb{Z}^d$. Recall from Proposition 3.1.3(ii) that d is even for $n \geq 3$ which is no restriction as discussed in Remark 3.1.4. We will show that $K_1(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)$ is not necessarily zero. For each $l \geq 0$, α induces an action $\Lambda^l(\alpha) : \mathbb{Z}_n \to \mathrm{Aut}(\Lambda^l \mathbb{Z}^d)$ of $\mathbb{Z}_n$ on the lth exterior power $\Lambda^l \mathbb{Z}^d$ of $\mathbb{Z}$-module $\mathbb{Z}^d$ as follows:

$$\begin{aligned} \Lambda^l(\alpha)(k)(x_1 \wedge \cdots \wedge x_l) &:= \Lambda^l(\alpha_k)(x_1 \wedge \cdots \wedge x_l) \\ &= \alpha_k(x_1) \wedge \cdots \wedge \alpha_k(x_l) \end{aligned}$$

for $k \in \mathbb{Z}_n$ and $x_1 \wedge \cdots \wedge x_l \in \Lambda^l \mathbb{Z}^d$. For notational convenience, we simply write $\Lambda(\alpha)$ for $\Lambda^l(\alpha)$. To compute $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$, we need to know the rank $\mathrm{rk}_{\mathbb{Z}}(\Lambda^l \mathbb{Z}^d)^{\mathbb{Z}_n}$ of the following submodule

$$(\Lambda^l \mathbb{Z}^d)^{\mathbb{Z}_n} := \{ f \in \Lambda^l \mathbb{Z}^d : \Lambda(\alpha_k)(f) = f \text{ for all } k \in \mathbb{Z}_n \}$$

of the fixed points.

Lemma 4.2.6. *Let $n \geq 3$ be an odd integer and $d := \phi(n)$. Then $\Lambda^d \mathbb{Z}^d = (\Lambda^d \mathbb{Z}^d)^{\mathbb{Z}_n}$.*

Proof. Since $\mathrm{rk}_{\mathbb{Z}} \Lambda^d \mathbb{Z}^d = 1$ and $\Lambda^d \mathbb{Z}^d$ is generated by $e_1 \wedge \cdots \wedge e_d$, it is enough to show that $\Lambda(\alpha_k)(e_1 \wedge \cdots \wedge e_d) = e_1 \wedge \cdots \wedge e_d$ for all $k \in \mathbb{Z}_n$. But this is obvious from

$$\begin{aligned} \Lambda(\alpha_k)(e_1 \wedge \cdots \wedge e_d) &= \alpha_k(e_1) \wedge \cdots \wedge \alpha_k(e_d) \\ &= C_n^k e_1 \wedge \cdots \wedge C_n^k e_d \\ &= \det(C_n^k) e_1 \wedge \cdots \wedge e_d \\ &= e_1 \wedge \cdots \wedge e_d, \end{aligned}$$

where the last equality comes from Proposition 3.1.3(iv). □

Lemma 4.2.7. *Let $n \geq 3$ and $d = \phi(n)$. Then we have*

$$C_n^{v-1} e_1 = e_v$$

for each $v = 1, \ldots, d$.

Proof. It is enough to note that the jth column of C_n is given by e_{j+1} for $j = 1, \ldots, d-1$. □

Lemma 4.2.8. *Suppose $\Lambda(\alpha) : \mathbb{Z}_n \to \mathrm{Aut}(\Lambda^l \mathbb{Z}^d)$ is an action of $\mathbb{Z}_n$ for some integers $n, l, d > 0$. Then for any element $x \in \Lambda^l \mathbb{Z}^d$, we have*

$$\sum_{k \in \mathbb{Z}_n} \Lambda(\alpha_k)(x) \in (\Lambda^l \mathbb{Z}^d)^{\mathbb{Z}_n}.$$

For two subsets I, J of $\mathbb{Z}$, set

$$J - I := \{ j - i \in \mathbb{Z} : i \in I, j \in J \}$$

and write $I \equiv J \pmod{n}$ if for each $i \in I$ there exists a $j \in J$ with $i - j \in n\mathbb{Z}$ and vice versa. As usual, $|I|$ denotes the cardinality of the set I.

Proposition 4.2.9. *Let $n \geq 3$ be an integer and $d := \phi(n)$. Consider the semidirect product $\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n$ with $\mathbb{Z}_n = \langle C_n \rangle$ where the conjugation action α is given by matrix multiplication as in* (4.1)*. If n satisfies the following condition* (A)*:*

(A) : the set $\{1, \ldots, d\}$ can be decomposed into a disjoint union $I \cup J$ with odd $|I|$ and odd $|J|$ satisfying $I - J \equiv \{1, 2, \ldots, n-1\} \pmod{n}$

then we have a nontrivial $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$.

Proof. Suppose $n \geq 3$ satisfies the condition (A), that is, the set $\{1, 2, \ldots, d\}$ can be divided into two disjoint sets $I = \{i_1, \ldots, i_l\}$ and $J = \{j_1, \ldots, j_{d-l}\}$ such that $l := |I|$ is odd (hence $|J|$ is odd) and

$$J - I \equiv \{1, 2, \ldots, n-1\} \ (\bmod\ n).$$

Then for any $k, t \in \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with $k \neq t$, there exist $i \in I$ and $j \in J$ such that $k + i \equiv t + j \pmod{n}$. So we have

$$\begin{aligned}
&\sum_{0 \leq k \neq t \leq n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \ldots \wedge e_{i_l}) \wedge \Lambda(\alpha_t)(e_{j_1} \wedge \ldots \wedge e_{j_{d-l}}) \\
= &\sum_{0 \leq k \neq t \leq n-1} C_n^k e_{i_1} \wedge \ldots \wedge C_n^k e_{i_l} \wedge C_n^t e_{j_1} \wedge \ldots \wedge C_n^t e_{j_{d-l}} \\
= &\sum_{0 \leq k \neq t \leq n-1} C_n^{k+i_1-1} e_1 \wedge \ldots \wedge C_n^{k+i_l-1} e_1 \wedge C_n^{t+j_1-1} e_1 \wedge \ldots \wedge C_n^{t+j_{d-l}-1} e_1 \\
= &\ 0,
\end{aligned}$$

where the second equality comes from Lemma 4.2.7. By Lemma 4.2.6, we have

$$\Lambda(\alpha_k)(e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}) = e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge e_{j_1} \wedge \cdots \wedge e_{j_{d-l}}$$

for all k. Thus

$$\begin{aligned}
0 \neq \; & n(e_{i_1} \wedge \ldots \wedge e_{i_l} \wedge e_{j_1} \wedge \ldots \wedge e_{j_{d-l}}) \\
= \; & \sum_{k=0}^{n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \ldots \wedge e_{i_l} \wedge e_{j_1} \wedge \ldots \wedge e_{j_{d-l}}) \\
= \; & \Big(\sum_{k=0}^{n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \ldots \wedge e_{i_l})\Big) \wedge \Big(\sum_{t=0}^{n-1} \Lambda(\alpha_t)(e_{j_1} \wedge \ldots \wedge e_{j_{d-l}})\Big) \\
& - \sum_{0 \leq k \neq t \leq n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \ldots \wedge e_{i_l}) \wedge \Lambda(\alpha_t)(e_{j_1} \wedge \ldots \wedge e_{j_{d-l}}) \\
= \; & \Big(\sum_{k=0}^{n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \ldots \wedge e_{i_l})\Big) \wedge \Big(\sum_{t=0}^{n-1} \Lambda(\alpha_t)(e_{j_1} \wedge \ldots \wedge e_{j_{d-l}})\Big),
\end{aligned}$$

and hence we have

$$\sum_{k=0}^{n-1} \Lambda(\alpha_k)(e_{i_1} \wedge \ldots \wedge e_{i_l}) \neq 0$$

and

$$\sum_{t=0}^{n-1} \Lambda(\alpha_t)(e_{j_1} \wedge \ldots \wedge e_{j_{d-l}}) \neq 0.$$

But these two elements belong to $(\Lambda^l \mathbb{Z}^d)^{\mathbb{Z}_n}$ and $(\Lambda^{d-l} \mathbb{Z}^d)^{\mathbb{Z}_n}$ respectively by Lemma 4.2.8, so that $\sum_{l \geq 0} \mathrm{rk}_{\mathbb{Z}}((\Lambda^{2l+1}\mathbb{Z}^d)^{\mathbb{Z}_n}) > 0$ follows. $\square$

Now we provide a condition equivalent to the condition (A) used in Proposition 4.2.9.

Proposition 4.2.10. *Let $n \geq 7$ be an odd number and $d := \phi(n)$. Then $2d \geq n + 5$ holds if and only if n satisfies the condition* (A) *of Proposition 4.2.9.*

Proof. For the direction 'only if', we take $I = \{1, 2, d\}$ and $J = \{3, 4, \ldots, d-$

$1\}$. Note that this choice of I and J is possible since $n \geq 7$. Then we have

$$\begin{aligned} J - I &\equiv \{1, 2, \ldots, d-2\} \cup \{n-d+3, n-d+4, \ldots, n-1\} \pmod{n} \\ &\equiv \{1, 2, \ldots, n-1\} \pmod{n}, \end{aligned}$$

which follows from the condition $2d \geq n+5$ (or $d-2 \geq n-d+3$).

For the converse, let $\{I, J\}$ be such a partition of $\{1, \ldots, d\}$ that $|I|$ and $|J|$ are odd and $J - I \equiv \{1, 2, \ldots, n-1\} \pmod{n}$. Assume $1 \in I$. Then the sets

$$P := (J - I) \cap \mathbb{N} \quad \text{and} \quad N := (J - I) \setminus P$$

contain the positive and negative integers of $J - I$ respectively. With $N' := \{n + m : m \in N\}$ of positive integers, one has $N' \equiv N (\bmod\ n)$. Also

$$P \subset \{1, 2, \ldots, d-1\} \quad \text{and} \quad N' \subset \{n + (1-d), n + (2-d), \ldots, n-1\} \tag{4.8}$$

is clear, hence $P \cup N' \subset \{1, 2, \ldots, n-1\}$ and $|P|, |N'| \leq d-1$. From

$$J - I = P \cup N \equiv P \cup N' \pmod{n} \quad \text{and} \quad J - I \equiv \{1, 2, \ldots, n-1\} \pmod{n},$$

it follows that

$$P \cup N' = \{1, 2, \ldots, n-1\} \tag{4.9}$$

since $P \cup N'$ has only positive integers less than n. Thus $n - 1 \leq |P| + |N'| \leq 2(d-1)$, namely $2d \geq n+1$ must hold. Since n is odd, we have $2d = n+1$, $2d = n+3$ or $2d \geq n+5$.

To show $2d \geq n+5$, first suppose $2d = n+1$. Then $2(d-1) = n-1$ so that we have $N' = \{n+1-d, n+2-d, \ldots, n-1\}$ and thus $1 - d \in N$. But $1 - d$ is not equal to any number $j - i$ for $j \in J$ and $i \in I$. Thus $2d \neq n+1$.

Now suppose that $2d = n+3$, that is, $2(d-1) = n+1$ holds. Set

$$L := \{1, 2, \ldots, d-1\} \quad \text{and} \quad R := \{n+1-d, n+2-d, \ldots, n-1\},$$

then $2d = n + 3$ is the case exactly when

$$L \cap R = \{d-2, d-1\} = \{n+1-d, n+2-d\}. \tag{4.10}$$

Also (4.9) implies that

$$\{1, 2, \ldots, d-3\} \subset P \quad \text{and } \{n+3-d, n+4-d, \ldots, n-1\} \subset N'. \tag{4.11}$$

Note here that if $m \in L \setminus R = \{1, \ldots, d-3\}$ then $m \in J - I$ and that if $m \in R \setminus L$ then $m - n \in J - I$. Moreover each $m \in L \cap R$ satisfies the following:

$$m \notin J - I \Rightarrow m - n \in J - I. \tag{4.12}$$

We claim that

$$\{1, d\} \subset I \quad \text{and} \quad \{2, d-1\} \subset J.$$

First observe that for $m := n + 1 - d \in L \cap R$, $m - n = 1 - d \notin J - I$ since $1 \in I$. By (4.12), $m = n + 1 - d \in J - I$ and hence $d - 2 \in J - I$ by (4.10). Thus we have

$$(d, 2) \in J \times I \quad \text{or} \quad (d-1, 1) \in J \times I. \tag{4.13}$$

Since $3 - d \in J - I$ by (4.11), we should have at least one of the following;

$$(3, d) \in J \times I, \ (2, d-1) \in J \times I, \quad \text{or} \quad (1, d-2) \in J \times I.$$

But $(2, d-1) \notin J \times I$ and $(1, d-2) \notin J \times I$ because of (4.13) and our assumption that $1 \in I$. Thus $(3, d) \in J \times I$. Since $d \notin J$ we also have $d - 1 \in J$ by (4.13). From (4.10) $d - 1 = n + 2 - d$ and thus we have $n + 2 - d = d - 1 \notin J - I$ (otherwise, $d \notin J$). Then by (4.12), $2 - d \in J - I$ which can occur only when $(2, d) \in J \times I$ or $(1, d-1) \in J \times I$. Since $1 \in I$, we obtain $2 \in J$. This proves the claim. Next we show that

$$\{1, d\} \subset I \quad \text{and} \quad \{2, 3, d-2, d-1\} \subset J$$

providing a proof that can be repeated until we reach the step $\{1, d\} = I$ and $\{2, \ldots, d-1\} = J$, where we meet a contradiction to the assumption that both I and J have odd number of elements. By (4.11), with $k = 2$, $n + (k+1) - d \in N'$ and $d - (k+1) \in P$. Thus

$$(k+1) - d \in J - I \text{ and } \ d - (k+1) \in J - I.$$

Note that $(k+1) - d = 3 - d \in J - I$ can happen only if $(k+1, d) = (3, d) \in J \times I$, $(k, d-1) = (2, d-1) \in J \times I$, or $(1, d-k) = (1, d-2) \in J \times I$. But obviously $(k+1, d) = (3, d) \in J \times I$ is the only possible case and we have $k+1 = 3 \in J$. Since $d-(k+1) = d-3 \in J-I$ we obtain $\{2, 3, d-2, d-1\} \in J$. (One can repeat the same argument on k.)

So far we have shown that the cases $2d = n+1$ and $2d = n+3$ should be excluded from $2d \geq n+1$, which gives $2d \geq n+5$ as desired. $\square$

Now we summarize Proposition 4.2.9 and Proposition 4.2.10 together with Theorem 4.2.1 to state the following result on a sufficient condition to have nonzero K_1-group of $C^*(\mathbb{Z}^d \rtimes \mathbb{Z}_n)$.

Theorem 4.2.11. *Let $n \geq 3$ be an integer with $d = \phi(n)$. Let $\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n$ be the semidirect product group by the conjugation action α of $\mathbb{Z}_n = \langle C_n \rangle$ as in* (4.1)*. If $2d \geq n+5$, then $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) = \mathbb{Z}^{s_1}$ for some integers $s_1 > 0$.*

Remark 4.2.12. In Theorem 4.2.11, for $3 \leq n \leq 6$, the condition $2d \geq n+5$ does not hold and in this case, in fact, corresponding K_1-group is trivial by Theorem 4.2.1. Also note that the condition $2d \geq n+5$ is equivalent to $K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \neq 0$ for prime n since a prime n satisfies $2d \geq n+5$ if and only if $n \geq 7$. Also we do not require anymore that n is odd since it is abundant. To see this, suppose that n is a positive even integer. There are two possibilities of the prime factorization of n. One is the case that there is a prime factor $p > 2$. In this case we can write

$$n = 2^{t_0} p_1^{t_1} \cdots p_r^{t_r}$$

where $r > 0$ is an integer and $2 < p_1 < \cdots < p_r$ are prime numbers and t_i's are positive integers. Then we have

$$\begin{aligned}(2\phi(n) - n)p_1 \cdots p_r &= [2^{t_0} p_1^{t_1-1} \cdots p_r^{t_r-1}(p_1 - 1) \cdots (p_r - 1) - n]p_1 \cdots p_r \\ &= n[(p_1 - 1) \cdots (p_r - 1) - p_1 \cdots p_r] \\ &< 0.\end{aligned}$$

The other case is that $n = 2^t$ for some integer $t > 0$. In this case it is easy to see that $2\phi(n) = n$. In any case we have that $2d \leq n$ if n is even, and hence $2d \geq n + 5$ is impossible.

4.2.3 Beyond C_n

As we have seen in the previous subsections, we could apply Theorem 4.2.1 to know whether $K_1(\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n) = K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$ is zero or not because the conjugation α of $\mathbb{Z}_n = \langle C_n \rangle$ is free outside the origin. But as we mentioned in Remark 3.2.3, not all the elements of finite order in $\mathrm{GL}_d(\mathbb{Z})$ are realized as C_n with $d = \phi(n)$. At this moment let us consider the matrix A of the form

$$A = \begin{pmatrix} C_{n_1} & & & \\ & C_{n_2} & & \\ & & \ddots & \\ & & & C_{n_r} \end{pmatrix} \tag{4.14}$$

where C_{n_i}'s are the companion matrices of the cyclotomic polynomials $\Phi_{n_i}(x)$. It is easy to see that the size of the matrix A is given by $d := \sum_i \phi(n_i)$ and the order of A is given by $n := \mathrm{lcm(n_1, \ldots, n_r)}$.

Proposition 4.2.13. *Let $A \in \mathrm{GL}_d(\mathbb{Z})$ be a matrix of finite order n which is of the form in* (4.14) *with $d := \sum_i \phi(n_i)$ and $n := \mathrm{lcm(n_1, \ldots, n_r)}$. Let α be the conjugation action of $\mathbb{Z}_n$ on $\mathbb{Z}^d$. Then α is free outside the origin if and only if $n_1 = \cdots = n_r$.*

Proof. For the direction of 'if' part, assume that $n_1 = \cdots = n_r$. Then A is a matrix of order n in $\mathrm{GL}_d(\mathbb{Z})$ which is of the form

$$A = \begin{pmatrix} C_n & & & \\ & C_n & & \\ & & \ddots & \\ & & & C_n \end{pmatrix}.$$

Put $d' = \phi(n)$. Suppose that the action of $\mathbb{Z}_n = \langle A \rangle$ on $\mathbb{Z}^d$ is not free outside the origin, that is, there exists $k \in \{1, \ldots, n-1\}$ satisfying

$$A^k x = x \tag{4.15}$$

for some nonzero $x \in \mathbb{Z}^d$. Write $x = (x_1, \cdots, x_r)$ with $x_i \in \mathbb{Z}^{d'}$. Then the equation (4.15) can be written in matrix form as follows:

$$\begin{pmatrix} C_n^k & & & \\ & C_n^k & & \\ & & \ddots & \\ & & & C_n^k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix}.$$

This implies that there exists $j \in \{1, \ldots, r\}$ such that $C_n^k x_j = x_j$ with nonzero $x_j \in \mathbb{Z}^{d'}$. But this is impossible since the action of $\langle C_n \rangle$ on $\mathbb{Z}^{d'}$ is free outside the origin by Proposition 4.2.3. So we proved the 'if' part.

For the other direction, without loss of generality we may assume that $n_1 \neq n_2$ with $n_1 < n_2$. Then clearly we have $n_1 < n$. Let 0_i denote the zero vector in $\mathbb{Z}^{\phi(n_i)}$ and take a nonzero vector $x_1 \in \mathbb{Z}^{\phi(n_1)}$ and put

$$x = \begin{pmatrix} x_1 \\ 0_2 \\ \vdots \\ 0_r \end{pmatrix}$$

which is a nonzero vector in $\mathbb{Z}^d$. Then we have

$$A^{n_1}x = \begin{pmatrix} C_{n_1}^{n_1} & & & \\ & C_{n_2}^{n_1} & & \\ & & \ddots & \\ & & & C_{n_r}^{n_1} \end{pmatrix} \begin{pmatrix} x_1 \\ 0_2 \\ \vdots \\ 0_r \end{pmatrix} = \begin{pmatrix} x_1 \\ 0_2 \\ \vdots \\ 0_r \end{pmatrix} = x.$$

This shows that the action of $\mathbb{Z}_n = \langle A \rangle$ on $\mathbb{Z}^d$ is not free outside the origin. This proves the 'only if' part. $\square$

Remark 4.2.14. Proposition 4.2.13 results in the following consequence: If we allow the cyclic group $\mathbb{Z}_n$ to be generated by a matrix A of the form in (4.14), then we can fully use the formulas for K_*-groups of $C^*(\mathbb{Z}^d \rtimes \mathbb{Z}_n)$ in Theorem 4.2.1 only for those d's which are multiples of $\phi(n)$. In this thesis, we deal only with the case of $d = \phi(n)$. The other cases of $d = k\phi(n)$ for some integer $k > 1$ are beyond our scope.

Chapter 5

Noncommutative simple tori on which finite cyclic groups act canonically

We have seen in the previous chapter that if α is a canonical action of $\mathbb{Z}_n = \langle C_n \rangle$ on a higher dimensional simple d-torus $\mathcal{A}_\Theta$, then we are well informed about the crossed products $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ since we know how to compute the decisive invariants, namely the K-groups $K_*(\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n)$ which are equal to $K_*(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n))$ by Theorem 4.1.11 and are not necessarily zero by Theorem 4.2.11. So it seems that many of the crossed products $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ are far from being AF. But what is not yet clear is if there does exist any d-dimensional noncommutative simple torus $\mathcal{A}_\Theta$ that actually admits the canonical action by the group $\mathbb{Z}_n$.

In this chapter we show that there do really exist such higher dimensional simple tori. With this we can clarify the existence of non-AF crossed products of higher dimensional simple tori by finite group actions. Also we go one step further to establish a general form of a nondegenerate Θ so that there exists a canonical action of $\mathbb{Z}_n$ on the simple torus $\mathcal{A}_\Theta$ in the case of prime n. In the end of the chapter we give some interesting examples.

5.1 Existence of noncommutative simple tori admitting canonical actions by finite cyclic groups

Recall from Notation 3.2.1 that

$$\mathcal{T}_{d,A}(\mathbb{R}) = \{\Theta \in \mathcal{T}_d(\mathbb{R}) : \Theta = A^t \Theta A\}$$

is the set of all skew symmetric matrices Θ such that the noncommutative tori $\mathcal{A}_\Theta$ admit the canonical action of the group $\langle A \rangle$ generated by A. For $A \in \mathrm{GL}_d(\mathbb{Z})$ and $\Theta \in \mathcal{T}_d(\mathbb{R})$, it is obvious that $A \in G_\Theta$ if and only if $\Theta \in \mathcal{T}_{d,A}(\mathbb{R})$.

Lemma 5.1.1. *Let A be a matrix in $\mathrm{GL}_d(\mathbb{Z})$ of finite order n. Then we have*

$$\mathcal{T}_{d,A}(\mathbb{R}) = \Big\{ \sum_{k=0}^{n-1} (A^k)^t \Theta A^k : \Theta \in \mathcal{T}_d(\mathbb{R}) \Big\}.$$

Proof. Since A is of order n, we see that any skew symmetric matrix $\Theta := \displaystyle\sum_{k=0}^{n-1} (A^k)^t \Theta' A^k$ (for $\Theta' \in \mathcal{T}_d(\mathbb{R})$) satisfies the condition $A^t \Theta A = \Theta$ to be an element of $\mathcal{T}_{d,A}(\mathbb{R})$. Conversely, if $\Theta \in \mathcal{T}_{d,A}(\mathbb{R})$, that is $A^t \Theta A = \Theta$, then $\Theta = \frac{1}{n} \sum_{k=0}^{n-1} (A^k)^t \Theta A^k$ is rather clear. □

Now we show that for each $n \geq 3$, the cyclic group $\mathbb{Z}_n$ acts canonically on some noncommutative simple $\phi(n)$-dimensional tori $\mathcal{A}_\Theta$. Note from the following theorem that since $\phi(n) = p_1^{r_1 - 1}(p_1 - 1) \cdots p_s^{r_s - 1}(p_s - 1)$ when $n = p_1^{r_1} \cdots p_s^{r_s}$ is the prime factorization of n, the group $\mathbb{Z}_n$ always acts on some noncommutative higher dimensional simple tori whenever $n = 5$ or $n \geq 7$.

Theorem 5.1.2. *Let $n \geq 3$ and $d := \phi(n)$. Then there exist simple d-dimensional tori $\mathcal{A}_\Theta$ on which the group $\mathbb{Z}_n = \langle C_n \rangle$ acts canonically.*

Proof. We show that $\mathcal{T}_{d,C_n}(\mathbb{R})$ contains nondegenerate matrices. Let θ be an irrational number. Then by Lemma 5.1.1,

$$\Theta := \theta \sum_{k=0}^{n-1} (C_n^k)^t (C_n^t - C_n) C_n^k$$

is a skew symmetric matrix in $\mathcal{T}_{d,C_n}(\mathbb{R})$. To show that Θ is nondegenerate, suppose $\Theta x \in \mathbb{Z}^d$ for some $x \in \mathbb{Z}^d$. Since the entries of C_n are integers and θ is irrational, we must have $\big(\sum_k (C_n^k)^t (C_n^t - C_n) C_n^k\big) x = 0$ in $\mathbb{Z}^d$. Then

$$\begin{aligned}
0 &= \Big(\sum_{k=0}^{n-1} (C_n^k)^t (C_n^t - C_n) C_n^k\Big) x \\
&= \Big(C_n^t \sum_{k=0}^{n-1} (C_n^k)^t C_n^k - \big(\sum_{k=0}^{n-1} (C_n^k)^t C_n^k\big) C_n\Big) x \\
&= \Big(C_n^t \sum_{k=0}^{n-1} (C_n^k)^t C_n^k - C_n^t \big(\sum_{k=0}^{n-1} (C_n^k)^t C_n^k\big) C_n^2\Big) x \\
&= C_n^t \sum_{k=0}^{n-1} (C_n^k)^t C_n^k \,(I_d - C_n^2)\, x,
\end{aligned}$$

where I_d is the $d \times d$ identity matrix. Since the matrix $\sum_k (C_n^k)^t C_n^k$ in the last equation above is of the form

$$\sum_k (C_n^k)^t C_n^k = I_d + (\text{positive}),$$

by the functional calculus, we see that $\sum_k (C_n^k)^t C_n^k$ is invertible in $\mathbb{M}_d(\mathbb{C})$. Thus $(I_d - C_n^2)x$ must be zero. But, as we have seen in Proposition 4.2.3, $x \neq C_n^2 x$ for nonzero $x \in \mathbb{Z}^d$ and $n \geq 3$, and we conclude that Θ is nondegenerate.

$\square$

We can finally address the following result obtained from combination of Proposition 4.2.4, Theorem 4.2.11 together with Theorem 5.1.2.

Theorem 5.1.3. *Let $n \geq 3$ be an integer with $d = \phi(n)$. Then there exists a canonical action $\alpha : \mathbb{Z}_n \to \mathrm{Aut}(\mathcal{A}_\Theta)$ of $\mathbb{Z}_n$ on a simple d-dimensional noncommutative torus $\mathcal{A}_\Theta$. If $\alpha : \mathbb{Z}_n \to \mathrm{Aut}(\mathcal{A}_\Theta)$ is a canonical action of the group $\mathbb{Z}_n$ generated by C_n on a simple d-dimensional torus $\mathcal{A}_\Theta$, then the crossed product $\mathcal{A}_\Theta \rtimes_\alpha \mathbb{Z}_n$ is an AT algebra. Furthermore,*

(i) *If $2d \geq n+5$ holds, then the crossed product is not an AF algebra.*

(ii) *If n is even, then the crossed product is an AF algebra.*

(iii) *If n is prime, the crossed product is an AF algebra if and only if $n = 3, 5$.*

We obtain the following which is a part of main result in [4] as a corollary of Theorem 5.1.3.

Corollary 5.1.4. *Let $\alpha : \mathbb{Z}_n \to \mathrm{Aut}(\mathcal{A}_\theta)$ $(n = 3, 4, 6)$ be the restricted actions of $\mathrm{SL}_2(\mathbb{Z})$ on an irrational rotation algebra $\mathcal{A}_\theta$ to its finite subgroups. Then the crossed products $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_n$ are all AF algebras.*

5.2 Noncommutative simple tori admitting canonical actions by finite cyclic groups of prime orders

In case that $n(\geq 3)$ is prime, we especially can find all the skew symmetric matrices $\Theta \in \mathcal{T}_{n-1}(\mathbb{R})$ such that $\mathcal{A}_\Theta$ admits the canonical action of the group

$\mathbb{Z}_n = \langle C_n \rangle$. We begin with finding the general form of Θ for which the cyclic group generated by a companion matrix C of finite order can possibly act on $\mathcal{A}_\Theta$.

Lemma 5.2.1. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ be a nonzero skew symmetric matrix and let $C \in \mathrm{GL}_d(\mathbb{Z})$ be the companion matrix of a monic polynomial with degree $d(\geq 3)$. Assume that C is of order n and $\mathcal{A}_\Theta$ admits the canonical action by $\mathbb{Z}_n = \langle C \rangle$, then Θ has the following form:*

$$\Theta = \begin{pmatrix} 0 & \theta_0 & \theta_1 & \theta_2 & \cdots & \theta_{d-3} & \theta_{d-2} \\ & 0 & \theta_0 & \theta_1 & \theta_2 & \cdots & \theta_{d-3} \\ & & 0 & \theta_0 & \theta_1 & \ddots & \vdots \\ & & & 0 & \theta_0 & \ddots & \theta_2 \\ & & & & 0 & \ddots & \theta_1 \\ & & & & & \ddots & \theta_0 \\ & & & & & & 0 \end{pmatrix} \tag{5.1}$$

for $\theta_i \in \mathbb{R}$, $i = 0, \ldots, d-2$.

Proof. Recall that $\mathbb{Z}_n = \langle C \rangle$ canonically acts on $\mathcal{A}_\Theta$ if and only if $C \in G_\Theta$, where $G_\Theta = \{A \in \mathrm{GL}_d(\mathbb{Z}) : A^t \Theta A = \Theta\}$. Let Θ_i and Θ^j be the ith row and jth column of $\Theta = (\theta_{ij})$, respectively for $i, j = 1, \ldots, d$. Let C be the companion matrix of a monic polynomial $a_1 + a_2 x + \cdots a_d x^{d-1} + x^d$ of degree d ($a_1 \neq 0$ because C is invertible). Then we can write C (see (3.1)) as the sum $C = A + B$ of two matrices A and B, where

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & 0 & a_2 \\ 0 & 0 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_d \end{pmatrix}.$$

Then a computation, with $\mathbf{a} = (a_1, \ldots, a_d)^t$, shows that

$$\begin{aligned} C^t \Theta C &= A^t \Theta A + A^t \Theta B + B^t \Theta A + B^t \Theta B \\ &= \begin{pmatrix} \theta_{22} & \cdots & \theta_{2d} & 0 \\ \vdots & \ddots & \vdots & 0 \\ \theta_{d2} & \cdots & \theta_{d,d} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & \Theta_2\,\mathbf{a} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \Theta_d\,\mathbf{a} \\ 0 & \cdots & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ (\Theta^2)^t\,\mathbf{a} & \cdots & (\Theta^d)^t\,\mathbf{a} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \mathbf{a}^t \Theta\,\mathbf{a} \end{pmatrix}. \end{aligned}$$

The assertion then follows from the fact that $C^t\Theta C$ is equal to Θ. $\square$

We know from Theorem 5.1.2 that for each $n \geq 3$ there exists a simple $\phi(n)$-dimensional torus $\mathcal{A}_\Theta$ which admits the canonical action of $\mathbb{Z}_n$ generated by the companion matrix C_n of the nth cyclotomic polynomial. In the following theorem, we show for prime $p \geq 3$ exactly when a $\phi(p)$-dimensional torus $\mathcal{A}_\Theta$ (not necessarily simple) admits such a canonical action of $\mathbb{Z}_p$ generated by C_p.

We consider the following skew symmetric matrices:

$$\Theta = \begin{pmatrix} 0 & \theta_0 & \theta_1 & \theta_2 & \cdots & -\theta_2 & -\theta_1 \\ & 0 & \theta_0 & \theta_1 & \theta_2 & \cdots & -\theta_2 \\ & & 0 & \theta_0 & \theta_1 & \ddots & \vdots \\ & & & 0 & \theta_0 & \ddots & \theta_2 \\ & & & & 0 & \ddots & \theta_1 \\ & & & & & \ddots & \theta_0 \\ & & & & & & 0 \end{pmatrix}. \tag{5.2}$$

Theorem 5.2.2. *Let $p \geq 3$ be a prime number, $d := \phi(p)(= p-1)$, and Θ a skew symmetric $d \times d$ matrix. Then $\mathcal{A}_\Theta$ admits a canonical action of $\mathbb{Z}_p = \langle C_p \rangle$, namely $C_p^t \Theta C_p = \Theta$, if and only if Θ is of the form in* (5.2)*. Moreover if Θ is of the form in* (5.2) *with the real numbers $1, \theta_0, \theta_1, \ldots, \theta_{(p-3)/2}$ that are independent over $\mathbb{Z}$, then $\mathcal{A}_\Theta$ is simple.*

Proof. First note that if Θ is a skew symmetric $d \times d$ matrix such that $\mathcal{A}_\Theta$ admits the canonical action of $\mathbb{Z}_p = \langle C_p \rangle$, then by Lemma 5.2.1, $\Theta = (\theta_{ij})$ must be of the form in (5.1). Since the pth cyclotomic polynomial is $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$, with $\mathbf{a} = (-1, \ldots, -1)^t$, one can repeat the computation performed in the proof of Lemma 5.2.1 to obtain that

$$\begin{aligned}
C_p^t \Theta C_p &= A^t\Theta A + A^t\Theta B + B^t\Theta A + B^t\Theta B \\
&= \begin{pmatrix} \theta_{22} & \cdots & \theta_{2d} & 0 \\ \vdots & \ddots & \vdots & 0 \\ \theta_{d2} & \cdots & \theta_{d,d} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & -\sum_j \theta_{2j} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\sum_j \theta_{dj} \\ 0 & \cdots & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$+ \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -\sum_i \theta_{i2} & \cdots & -\sum_i \theta_{id} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \sum_{i,j=1}^d \theta_{ij} \end{pmatrix}.$$

But then from the fact that $C_p^t\Theta C_p$ is equal to the matrix

$$\Theta = \begin{pmatrix} 0 & \theta_{12} & \theta_{13} & \cdots & \theta_{1,p-2} & \theta_{1,p-1} \\ -\theta_{12} & 0 & \theta_{12} & \theta_{13} & \cdots & \theta_{1,p-2} \\ -\theta_{13} & -\theta_{12} & 0 & \theta_{12} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \theta_{13} \\ -\theta_{1,p-2} & & & & 0 & \theta_{12} \\ -\theta_{1,p-1} & -\theta_{1,p-2} & \cdots & -\theta_{13} & -\theta_{12} & 0 \end{pmatrix},$$

comparing the last columns of $C_p^t\Theta C_p$ and Θ, we have

$$\theta_{1,p-1} = -\sum_j \theta_{2j},\ \theta_{1,p-2} = -\sum_j \theta_{3j},\ \ldots,\ \theta_{13} = -\sum_j \theta_{d-1,j},\ \theta_{12} = -\sum_j \theta_{d,j}.$$

Note here that for any Θ of the above form, $\sum_j \theta_{kj} + \sum_j \theta_{d-(k-1),j} = 0$ for each $k = 1, \ldots, d/2$, which shows

$$\theta_{1,p-1} = -\sum_j \theta_{2j} = \sum_j \theta_{d-1,j} = -\theta_{13}.$$

Similarly we see that $\theta_{1k} + \theta_{1,d-(k-1)} = 0$ for all $k = 3, \ldots, d/2$. Conversely, if Θ is in the form of (5.2), it is just a simple observation from the above computation that $C_p^t \Theta C_p = \Theta$ holds.

If the real numbers $1, \theta_0, \theta_1, \ldots, \theta_{(p-3)/2}$ are independent over $\mathbb{Z}$, it is not hard to see that the skew symmetric matrix Θ is nondegenerate, which proves the last assertion. $\square$

5.3 Canonical actions on 4-dimensional simple tori $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$

In this section, we examine concrete examples of canonical actions on 4-dimensional tori of the form $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$, that is, 2-fold tensor product of rotation algebra $\mathcal{A}_\theta$ with itself. Taking n-fold tensor product seems to be the easiest way to construct higher dimensional noncommutative tori.

Recall from Example 3.2.2 that $\phi(n) = 4$ if and only if $n = 5, 8, 10, 12$ and that for those n, $\Theta \in \mathcal{T}_{4,C_n}(\mathbb{R})$ has the form

$$\mathcal{T}_{4,C_n}(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \theta & \mu & \nu_n \\ & 0 & \theta & \mu \\ & & 0 & \theta \\ & & & 0 \end{pmatrix} : \theta, \mu \in \mathbb{R} \right\}, \tag{5.3}$$

where $\nu_5 = -\mu$, $\nu_8 = \theta$, $\nu_{10} = \mu$, and $\nu_{12} = 2\theta$. Moreover, $\Theta \in \mathcal{T}_{4,C_n}(\mathbb{R})$ is easily seen to be nondegenerate whenever 1, θ, μ are independent over $\mathbb{Z}$.

Remark 5.3.1. Let $m = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_2}$ be the prime factorization of an integer $m \in \mathbb{N}$, where $p_1 < p_2 < \cdots < p_t$ are primes. Then it is known ([13, Theorem 2.7]) that the group $\mathrm{GL}_n(\mathbb{Z})$ has an element of order m if and only if

(1) $\sum_{i=1}^{t}(p_i - 1)p_i^{k_i-1} - 1 \leq n$ for $p_1^{k_1} = 2$, or

(2) $\sum_{i=1}^{t}(p_i - 1)p_i^{k_1-1} \leq n$ otherwise.

Thus, with $n = 4$, we see that any possible finite order of a matrix in $\mathrm{GL}_4(\mathbb{Z}) \setminus \{I_4\}$ is one of $2, 3, 4, 5, 6, 8, 10, 12$. It should be noted that the action by $\mathbb{Z}_5$ or $\mathbb{Z}_{10}$ is not conjugate to any product action (on $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$) of two canonical actions by finite cyclic subgroups $F(\subset \mathrm{SL}_2(\mathbb{Z}))$ on $\mathcal{A}_\theta$ because F is necessarily isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$, or $\mathbb{Z}_6$. So any product action of them yields an action of order $2, 3, 4, 6, 8$ or 12. It seems that people have not considered any possibility so far that there exists an action on $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ by finite group of order 5 or 10.

Now we consider 4-dimensional noncommutative tori that are isomorphic to the tensor product $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ of an irrational rotation algebra $\mathcal{A}_\theta$ with itself. If $\mathcal{A}_\Theta$ is associated with the following skew symmetric matrix

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}, \tag{5.4}$$

it is easily seen to be isomorphic to the tensor product $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$. Moreover the following skew symmetric matrices $\Theta_{n,\theta}$,

$$\Theta_{5,\theta} = \Theta_{10,\theta} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & \theta & 0 \\ 0 & -\theta & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} \in \mathcal{T}_{4,C_5}(\mathbb{R}) \cap \mathcal{T}_{4,C_{10}}(\mathbb{R}), \tag{5.5}$$

$$\Theta_{8,\theta} = \Theta_{12,\theta} = \begin{pmatrix} 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & \theta \\ -\theta & 0 & 0 & 0 \\ 0 & -\theta & 0 & 0 \end{pmatrix} \in \mathcal{T}_{4,C_8}(\mathbb{R}) \cap \mathcal{T}_{4,C_{12}}(\mathbb{R}), \tag{5.6}$$

(see (5.3)) give rise to 4-dimensional tori isomorphic to $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$:

Lemma 5.3.2. *Let Θ be the matrix in* (5.4) *with $\theta \in \mathbb{R}$ and $\Theta_{n,\theta}$ be one of the matrices in* (5.5) *or* (5.6) *for $n = 5, 8, 10, 12$. We then have the following:*

(i) *There exists $B_n \in \mathrm{GL}_n(\mathbb{Z})$ with $B_n^t \Theta_{n,\theta} B_n = \Theta$.*

(ii) *$G_\Theta = B_n^{-1} G_{\Theta_{n,\theta}} B_n$.*

(iii) *$\mathcal{A}_{\Theta_{n,\theta}}$ is isomorphic to $\mathcal{A}_\Theta$.*

Proof. (i) The following B_n is the desired matrix for each n:

$$B_5 = B_{10} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad B_8 = B_{12} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(ii) and (iii) then follow from Proposition 2.3.8 and its proof. □

Since, in the above situation, $C_n \in G_{\Theta_{n,\theta}}$, for $\theta \in \mathbb{R}$ and $n = 5, 8, 10, 12$, and

$$C \mapsto (B_n)^{-1} C B_n : G_{\Theta_{n,\theta}} \to G_\Theta$$

is a group isomorphism, we see that

$$A_n := (B_n)^{-1} C_n B_n$$

are the matrices (acting on $\mathcal{A}_\Theta$) of order n for $n = 5, 8, 10, 12$ by Lemma 5.3.2(ii), and actually given by

$$\begin{aligned} A_5 &= \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad A_8 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ A_{10} &= \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{5.7}$$

Proposition 5.3.3. *Every 4-dimensional noncommutative torus of the form $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ admits a canonical action α_n of $\mathbb{Z}_n = \langle A_n \rangle$ in (5.7) for $n = 5, 8, 10, 12$. More precisely, if $\mathcal{A}_\theta \otimes \mathcal{A}_\theta = C^*(u_1, u_2) \otimes C^*(u_3, u_4)$ is generated by 4 unitaries u_i's satisfying $u_2 u_1 = \exp(2\pi i\theta) u_1 u_2$, $u_4 u_3 = \exp(2\pi i\theta) u_3 u_4$, and $u_k u_l = u_l u_k$ for $k = 1, 2$ and $l = 3, 4$, we have the following:*

(i) $\alpha_5 : \mathbb{Z}_5 \to \mathrm{Aut}(\mathcal{A}_\theta \otimes \mathcal{A}_\theta)$ *is induced by the automorphism;*

$$\begin{aligned} &u_1 \mapsto u_2 u_4^*, \;\; u_2 \mapsto \exp(\pi i\theta) u_1^* u_2^*, \\ &u_3 \mapsto u_4, \qquad u_4 \mapsto \exp(\pi i\theta) u_1^* u_2^* u_3^*. \end{aligned} \tag{5.8}$$

(ii) $\alpha_8 : \mathbb{Z}_8 \to \mathrm{Aut}(\mathcal{A}_\theta \otimes \mathcal{A}_\theta)$ *is induced by the automorphism;*

$$\begin{aligned} &u_1 \mapsto u_3, \quad u_2 \mapsto u_4, \\ &u_3 \mapsto u_2, \quad u_4 \mapsto u_1^*. \end{aligned} \tag{5.9}$$

(iii) $\alpha_{10} : \mathbb{Z}_{10} \to \mathrm{Aut}(\mathcal{A}_\theta \otimes \mathcal{A}_\theta)$ *is induced by the automorphism;*

$$\begin{aligned} u_1 &\mapsto u_2 u_4^*, \quad u_2 \mapsto \exp(-\pi i\theta) u_1^* u_2, \\ u_3 &\mapsto u_4, \qquad u_4 \mapsto \exp(-\pi i\theta) u_1^* u_2 u_3^*. \end{aligned} \tag{5.10}$$

(iv) $\alpha_{12} : \mathbb{Z}_{12} \to \mathrm{Aut}(\mathcal{A}_\theta \otimes \mathcal{A}_\theta)$ *is induced by the automorphism;*

$$\begin{aligned} u_1 &\mapsto u_3, \quad u_2 \mapsto u_4, \\ u_3 &\mapsto u_2, \quad u_4 \mapsto \exp(-\pi i\theta) u_1^* u_2. \end{aligned} \tag{5.11}$$

Proof. We only show the case (i) here because the rest can be done similarly. Since $\mathcal{A}_\Theta$ is the twisted group algebra $C^*(\mathbb{Z}^4, \omega_\Theta) = C^*\{l_\Theta(e_j) : 1 \le j \le 4\}$ with the identification $u_j := l_\Theta(e_j)$ for each j, the action α_5 of $\mathbb{Z}_5 = \langle A_5 \rangle$ is determined by $\alpha_5(k)(l_\Theta(e_j))$ for $k \in \mathbb{Z}_5$ and $1 \le j \le 4$. For convenience, we write simply α for $\alpha_5(1)$ to have:

$$\begin{aligned} \alpha(l_\Theta(e_1)) &= l_\Theta(A_5 e_1) = l_\Theta(e_2 - e_4) \\ &= \overline{\omega_\Theta(e_2, -e_4)}\, l_\Theta(e_2) l_\Theta(e_4)^* \\ &= \exp(-\pi i \langle \Theta e_2, -e_4 \rangle)\, l_\Theta(e_2) l_\Theta(e_4)^* \\ &= l_\Theta(e_2) l_\Theta(e_4)^*, \\ \alpha(l_\Theta(e_2)) &= l_\Theta(A_5 e_2) = \exp(\pi i\theta) l_\Theta(e_1)^* l_\Theta(e_2)^*, \\ \alpha(l_\Theta(e_3)) &= l_\Theta(A_5 e_3) = l_\Theta(e_4), \\ \alpha(l_\Theta(e_4)) &= l_\Theta(A_5 e_4) = \exp(\pi i\theta) l_\Theta(e_1)^* l_\Theta(e_2)^* l_\Theta(e_3)^*. \end{aligned}$$

Thus α_5 is the action of $\mathbb{Z}_5$ generated by the automorphism α sending u_1 to $u_2 u_4^*$ and so on as stated in (5.8). $\square$

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국문초록

본 논문에서는 2 차원 비가환 원환면에 대한 정수환 위에서의 2 차 특수선형군의 군작용을 연장하여 고차원 비가환 원환체에 대한 정준 군작용을 연구한다. 본 논문에서는 유한군의 정준 군작용에 의한 고차원 비가환 단순 원환체의 교차곱은 AF 대수가 아닌 AT 대수가 될 수 있음을 보인다. 이는 2 차원 비가환 단순 원환체의 유한군에 의한 정준 군작용으로부터 얻어지는 교차곱은 모두 AF 대수라는 기존 결과가 고차원으로는 확장되지 않는다는 것을 의미한다.

주요어휘: 비가환 원환체, 군작용, C^*-교차곱, AF 대수
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